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Phil. Trans. R. Soc. Lond. A 1993 **345**, 545-576
doi: 10.1098/rsta.1993.0144

Accuracy of the generalized self-consistent method in modelling the elastic behaviour of periodic composites†

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Contents

Local stress and strain fields in the unit cell of an infinite, two-dimensional, periodic fibrous lattice have been determined by an integral equation approach. The effect of the fibres is assimilated to an infinite two-dimensional array of fictitious body forces in the matrix constituent phase of the unit cell. By subtracting a volume averaged strain polarization term from the integral equation we effectively embed a finite number of unit cells in a homogenized medium in which the overall stress and strain correspond to the volume averaged stress and strain of the constrained unit cell. This paper demonstrates that the zeroth term in the governing integral equation expansion, which embeds one unit cell in the homogenized medium, corresponds to the generalized self-consistent approximation. By comparing the zeroth term approximation with higher order approximations to the integral equation summation, both the accuracy of the generalized self-consistent composite model and the rate of convergence of the integral summation can be assessed. Two example composites are studied. For a tungsten/copper elastic fibrous composite the generalized self-consistent model is shown to provide accurate, effective, elastic moduli and local field representations. The local elastic transverse stress field within the representative volume element of the generalized self-consistent method is shown to be in error by much larger amounts for

† This paper was produced from the authors' disk by using the TEX typesetting system.

Phil. Trans. R. Soc. Lond. A (1993) **345**, 545–576
Printed in Great Britain

a composite with periodically distributed voids, but homogenization leads to a cancelling of errors, and the effective transverse Young's modulus of the voided composite is shown to be in error by only 23% at a void volume fraction of 75%.

1. Introduction

In a recent paper Christensen (1990) compared the behaviour of the differential method (Roscoe 1952), the Mori-Tanaka method (Benveniste 1987), and the generalized self-consistent method (Christensen & Lo 1979, 1986) when applied to the evaluation of the effective elastic moduli of composite materials. The effective shear modulus/viscosity of a polydisperse emulsion of solid spherical droplets was computed by the three methods and compared with the experimental work of Eilers (1941). Christensen showed that the prediction from the generalized self-consistent method correlates the experimental data of Eilers, while the computations from the differential and Mori-Tanaka methods underpredict the effective viscosity/shear modulus of the composite solid/solution by up to a factor of ten at 70% volume fraction. It is therefore of interest to compare the results of the generalized self-consistent composite model with those of a periodic composite, for which exact results can be obtained. By so doing, we are able to assess—for the first time—the accuracy of the generalized self-consistent method through a comparison with a method on equal footing.

The generalized self-consistent method in this paper differs from that derived by Christensen and Lo (1979, 1986) in several important ways. In their model the local elastic fields in the generalized self-consistent method for fibrous composites are usually computed for a representative volume element in which a cylindrical fibre is embedded in an annular matrix. This volume element is, in turn, embedded in an infinite effective elastic medium possessing transversely isotropic properties. They assume that the composite medium is comprised of representative volume elements of widely varying size, so that the overall elastic properties of the composite are well represented by a transversely isotropic medium surrounding the representative volume element. In their model the elastic fields, which vary throughout the fibre and matrix phases in the volume element, are computed exactly from an elasticity problem in which the representative volume element is constrained by the transversely isotropic homogenized medium. The overall elastic moduli of the homogenized medium are then obtained by volume averaging the constrained elastic stress and strain fields over the representative volume element.

In the present paper we introduce a generalized self-consistent method in which the representative volume element has a rectangular boundary. This represents one unit cell of an infinite two-dimensional periodic lattice. To obtain the generalized self-consistent approximation the remainder of the periodic lattice is then replaced by a homogenized medium possessing tetragonal elastic symmetry. The task which then confronts us is to determine the elastic stress and strain fields in the two composite phases within the unit cell when constrained by (*a*) the homogenized medium with tetragonal elastic symmetry; and (*b*) the remainder of the infinite periodic lattice. In both cases the elastic fields within the unit cell are determined numerically by solving a periodic boundary value problem for an

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infinite lattice of unit cells arranged in rectangular order. The elastic disturbance caused by the fibres in the matrix is treated as a fictitious body force problem and the net effect within the unit cell is determined by summation over all the fibres in the periodic lattice from an integral equation using a Green's function method.

In the numerical problem only the effects from a finite number of fibres surrounding the given unit cell are summed. This means that the medium which surrounds the outermost fibres corresponds to the matrix constituent phase of the unit cell. In order to obtain the correct elastic response within the unit cell, the outer medium, surrounding the finite number of fibres, should correspond to a homogenized representation of the remainder of the lattice rather than to the matrix constituent phase. This correspondence can be achieved by applying suitable surface tractions on the boundary separating the outer fibres from the matrix constituent phase. We show that the additional strain within the unit cell, which is caused by adding this layer of surface traction, is equivalent to subtracting a volume averaged polarization strain term from the original integral equation. This volume averaged term performs three functions in the Green's function integral equation. Its first office is to secure the equality of the overall uniform strain in the outer medium with the volume averaged strain of the constrained unit cell. Secondly, this volume averaged term is also necessary to convert the conditionally convergent (or divergent) integral equation into an absolutely convergent solution of the periodic boundary value problem. Finally, we find that this term effectively embeds a finite number of unit cells in the effective homogenized medium, and the generalized self-consistent approximation then emerges by considering only one term in the body force summation. In this approximation only one unit cell is embedded in the homogenized outer medium and the elastic symmetry of this medium is determined (not assumed) from the integral equation boundary value problem.

In the usual form of the generalized self-consistent method, the homogenized outer medium will possess the elastic symmetry of the volume averaged unit cell. If we elect to solve the elasticity problem of a unit cell embedded in this homogenized medium by a Green's function technique, then the normal process would involve constructing a Green's function which possesses the elastic symmetry of the homogenized medium, and then assimilating the embedded unit cell to a fictitious body force. The present method uses an isotropic Green's function appropriate to the matrix material, and the homogenized outer medium is simulated by applying a layer of surface traction to the surface separating the finite number of summed fibres from the actual surrounding matrix constituent phase.

The object of the present paper is to determine the variation of the local elastic stress field within the unit cell of a periodic composite, and to determine the errors in this stress field and the effective transverse Young's modulus when the remainder of the periodic lattice is replaced by a uniform homogenized material possessing tetragonal elastic symmetry.

2. Local stress and strain fields

Previous articles on Fourier's series and Green's function approaches relevant to the present work may be found in (Nemat-Nasser & Taya 1981; Nemat-Nasser *et al.* 1982; Nemat-Nasser & Iwakuma 1983; Iwakuma & Nemat-Nasser 1983;

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Nunan & Keller 1984; Accorsi & Nemat-Nasser 1986; Nemat-Nasser *et al.* 1986) and (Eshelby 1957; Barnett & Swanger 1971; Barnett 1972; Korringa 1973; Zeller & Dederichs 1973; Gubernatis & Krumhansl 1975; Levin 1976; Chen & Young 1977; Cleary *et al.* 1980; Mura 1987), respectively. A constitutive formulation given in previous papers (Walker *et al.* 1989, 1990, 1991, 1992), which is suitable for describing the inelastic behaviour of composite materials under thermomechanical loading conditions, is now specialized to isothermal elastic deformation. The equations are left in incremental form to permit convenient comparison with the preceding papers.

The form of Hooke's law suitable for infinitesimal strains can be written in the incremental form

$$\Delta\sigma_{ij}(\boldsymbol{r}) = D_{ijkl}(\boldsymbol{r})\,\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}), \tag{2.1}$$

where at the point in the unit cell with position vector $\boldsymbol{r}$, $\Delta\sigma_{ij}(\boldsymbol{r})$ is the local stress increment, $D_{ijkl}(\boldsymbol{r})$ is the elasticity tensor and $\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r})$ is the total strain increment. When a uniform overall strain increment of $\Delta\varepsilon^{0}_{kl}$ is applied to the periodic fibrous composite depicted in figure 1 *b* or *c*, the total strain increment can be written in the form

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} + \Delta p^{*}_{kl}(\boldsymbol{r}), \tag{2.2}$$

where the perturbation strain increment, $\Delta p^{*}_{kl}(\boldsymbol{r})$, represents the deviation from the overall strain increment, $\Delta\varepsilon^{0}_{kl}$, due to the presence of the fibres. The variation of this local field quantity is often determined (Lene 1986; Bahei-El-Din *et al.* 1987; Teply & Dvorak 1988; Bahei-El-Din & Dvorak 1989; Dvorak 1991) by applying periodic boundary conditions to a finite element discretization of the unit cell, or by breaking the unit cell into rectangular subvolumes (Aboudi 1989, 1991; Paley & Aboudi 1992) and enforcing periodicity through continuity of displacements and surface tractions at the unit cell boundary.

In the present paper the local field quantities are determined by discretizing the unit cell into rectangular subvolumes, and periodicity is enforced naturally by expanding the perturbation strain increment into a Fourier series. By applying the Poisson sum formula to the Fourier series representation we have shown (Walker *et al.* 1989, 1990) that the Green's function method is recovered. The Green's function method involves a summation of fictitious body forces due to the presence of fibres in the matrix constituent phase of the composite. Within the unit cell the local variation of the perturbation strain increment is evaluated by summing the contributions from these fictitious body forces in each fibre of the infinite periodic lattice. We will show in the sequel that the zeroth term of the infinite sum in the Green's function formulation corresponds to the generalized self-consistent method.

The elasticity tensor at any point $\boldsymbol{r}$ in the composite material may be written in the form

$$D_{ijkl}(\boldsymbol{r}) = D^{\mathrm{m}}_{ijkl} + \delta D_{ijkl}(\boldsymbol{r}), \tag{2.3}$$

where

$$\delta D_{ijkl}(\boldsymbol{r}) = \vartheta(\boldsymbol{r})\,(D^{\mathrm{f}}_{ijkl} - D^{\mathrm{m}}_{ijkl}). \tag{2.4}$$

In this relationship $\vartheta(\boldsymbol{r}) = 1$ in the fibre and $\vartheta(\boldsymbol{r}) = 0$ in the matrix, with D^{f}_{ijkl} denoting the elasticity tensor of the fibre and D^{m}_{ijkl} that of the matrix. We have found (Walker *et al.* 1989, 1990, 1991, 1992) that in the Fourier series approach the total strain increment in the unit cell of a three-dimensional periodic lattice

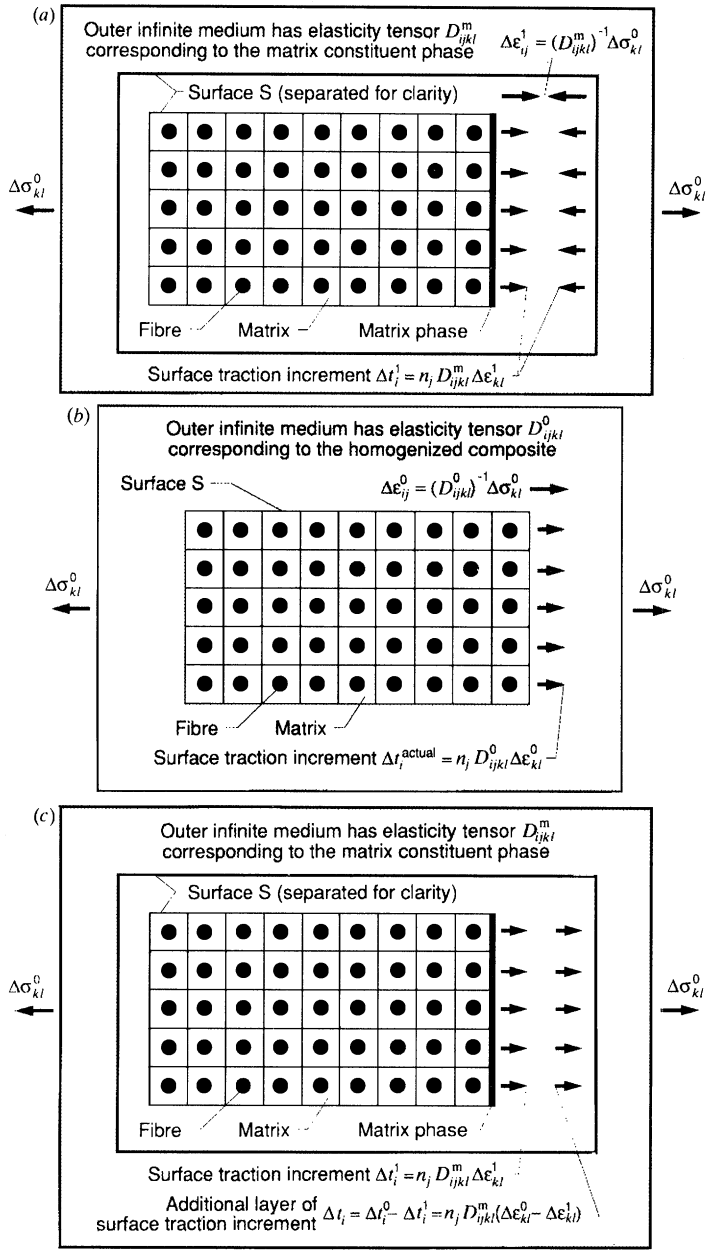

Figure 1. (*a*) A finite lattice embedded in an infinite outer medium whose elastic properties correspond to the matrix constituent phase. The strain increment inside S is governed by the integral equation (2.17). (*b*) A finite lattice embedded in an infinite outer medium whose elastic properties correspond to the homogenized (volume averaged) composite. The strain increment inside S is governed by the integral equation (2.22). The strain increment $\Delta\varepsilon^0_{kl}$ produces a surface traction increment $\Delta t_i^{\text{actual}} = n_j D^0_{ijkl}\Delta\varepsilon^0_{kl}$ on the surface S in the homogenized medium, but would produce a surface traction increment $\Delta t^0_i = n_j D^m_{ijkl}\Delta\varepsilon^0_{kl}$ when applied to the surface S in (*a*). (*c*) An additional layer of incremental surface traction applied to the surface, S, effectively embeds the unit cells in a homogenized medium. The strain increment inside S is then governed by the integral equation (2.22). The Green's function corresponds to the isotropic matrix phase and the effect of the outer homogenized medium is assimilated to the layer of surface traction on S.

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is determined by solving the integral equation,

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} - \frac{1}{V_{\mathrm{c}}}\sum\sum_{n_p=0}^{\pm\infty}{\sum}' g_{klmn}(\boldsymbol{\zeta})\iiint_{V_{\mathrm{c}}} \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'), \tag{2.5}$$

where the fourth rank tensor $g_{klmn}(\boldsymbol{\zeta})$ is given by

$$g_{klmn}(\boldsymbol{\zeta}) = \tfrac{1}{2}(\zeta_n\zeta_l M^{-1}_{mk}(\boldsymbol{\zeta}) + \zeta_n\zeta_k M^{-1}_{ml}(\boldsymbol{\zeta})), \tag{2.6}$$

in which the Christoffel stiffness tensor $M_{ij}(\boldsymbol{\zeta})$, with inverse $M^{-1}_{ij}(\boldsymbol{\zeta})$, is defined (Barnett & Swanger 1971; Barnett 1972; Mura 1987) by the relation

$$M_{ij}(\boldsymbol{\zeta}) = D^{\mathrm{m}}_{pijq}\zeta_p\zeta_q, \tag{2.7}$$

in which $\zeta_p = \xi_p/\sqrt{(\xi_m\xi_m)} = \xi_p/\xi$ are the components of a unit vector in the direction of the Fourier wave vector $\boldsymbol{\xi}$, with $\xi = \sqrt{(\xi_m\xi_m)}$ denoting its magnitude. In equation (2.5) the sum is taken over integer values in which

$$\xi_1 = \frac{2\pi n_1}{L_1}, \quad \xi_2 = \frac{2\pi n_2}{L_2}, \quad \xi_3 = \frac{2\pi n_3}{L_3} \tag{2.8}$$

and L_1, L_2, L_3 are the dimensions of the unit periodic cell in the x_1, x_2, x_3 directions, so that the volume of the unit cell is given by the relation, $V_{\mathrm{c}} = L_1L_2L_3$. The values of n_1, n_2, n_3 are given by

$$n_p = 0, \pm1, \pm2, \pm3, \ldots, \text{etc.}, \quad \text{for} \quad p = 1, 2, 3 \tag{2.9}$$

and the prime on the triple summation signs indicates that the term with $n_1 = n_2 = n_3 = 0$ is excluded from the sum.

In equation (2.5) the term containing the primed triple Fourier sum represents the perturbation strain increment in equation (2.2), so that if the Fourier coefficients in (2.5) are denoted by $A_{kl}(n_1, n_2, n_3)$, then

$$A_{kl}(n_1, n_2, n_3) = -\frac{1}{V_{\mathrm{c}}}\,g_{klmn}(\boldsymbol{\zeta})\iiint_{V_{\mathrm{c}}} \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}'}\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'). \tag{2.10}$$

We now define the polarization strain increment, $\Delta p_{kl}(\boldsymbol{r})$, to be the fully summed Fourier expansion of the perturbation strain increment, $\Delta p^{*}_{kl}(\boldsymbol{r})$, which includes the zeroth (constant) term, so that

$$\Delta p_{kl}(\boldsymbol{r}) = \sum\sum_{n_p=0}^{\pm\infty}\sum A_{kl}(n_1, n_2, n_3)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}} \tag{2.11}$$

and

$$\begin{aligned}\Delta p^{*}_{kl}(\boldsymbol{r}) &= \sum\sum_{n_p=0}^{\pm\infty}{\sum}' A_{kl}(n_1, n_2, n_3)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}} \\ &= \sum\sum_{n_p=0}^{\pm\infty}\sum A_{kl}(n_1, n_2, n_3)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{r}} - A_{kl}(0, 0, 0) \\ &= \Delta p_{kl}(\boldsymbol{r}) - A_{kl}(0, 0, 0),\end{aligned} \tag{2.12}$$

where the primed series in (2.5) is now written as the fully summed series minus the zeroth term. Now the zeroth term of a Fourier series is just the volume average of the function—whose Fourier expansion is required—over the unit cell of the periodic lattice, so that in equation (2.11)

$$A_{kl}\,(0,0,0) = \frac{1}{V_c}\iiint_{V_c} \Delta p_{kl}\,(\boldsymbol{r})\;\mathrm{d}V(\boldsymbol{r}) = \langle \Delta p_{kl}\,(\boldsymbol{r})\rangle, \tag{2.13}$$

where the angle brackets are used to denote the volume average of the enclosed expression over the unit cell. Accordingly, if we write the perturbation strain increment in terms of the polarization strain increment in the form

$$\Delta p^{*}_{kl}\,(\boldsymbol{r}) = \Delta p_{kl}\,(\boldsymbol{r}) - \langle \Delta p_{kl}\,(\boldsymbol{r})\rangle, \tag{2.14}$$

then the total strain increment can be written as

$$\Delta \varepsilon^{\mathrm{T}}_{kl}\,(\boldsymbol{r}) = \Delta \varepsilon^{0}_{kl} + \Delta p_{kl}\,(\boldsymbol{r}) - \langle \Delta p_{kl}\,(\boldsymbol{r})\rangle. \tag{2.15}$$

On volume averaging this equation over the unit cell we obtain

$$\left\langle \Delta \varepsilon^{\mathrm{T}}_{kl}\,(\boldsymbol{r})\right\rangle = \Delta \varepsilon^{0}_{kl}, \tag{2.16}$$

so that the omission of the zeroth term in the Fourier series representation of the perturbation strain increment automatically ensures that the volume average of the total strain increment, $\langle \Delta \varepsilon^{\mathrm{T}}_{kl}\,(\boldsymbol{r})\rangle$, over the unit cell (and hence by periodicity over the entire composite) is equal to the overall applied strain increment, $\Delta \varepsilon^{0}_{kl}$. It also ensures that the volume averaged perturbation strain increment, $\langle \Delta p^{*}_{kl}\,(\boldsymbol{r})\rangle$, is zero so that on average the strain field in the composite fluctuates as much above the overall applied strain field as it does below it.

If we assume that the effect of the cylindrical fibres can be assimilated to fictitious body forces distributed throughout each cylinder in the periodic lattice, then the total strain increment at any point within the unit cell in the Green's function formulation (Walker *et al.* 1989, 1990) is governed by the integral equation

$$\Delta \varepsilon^{\mathrm{T}}_{kl}\,(\boldsymbol{r}) = \Delta \varepsilon^{0}_{kl} - \iiint_{V} U_{klmn}\,(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}\,(\boldsymbol{r}')\,\Delta \varepsilon^{\mathrm{T}}_{rs}\,(\boldsymbol{r}')\;\mathrm{d}V\,(\boldsymbol{r}')\,, \tag{2.17}$$

where the fourth rank tensor $U_{klmn}\,(\boldsymbol{r}-\boldsymbol{r}')$ gives the kl component of the total strain increment at point $\boldsymbol{r}$ due to the mn component of a stress increment applied at point $\boldsymbol{r}'$ in the infinite matrix with elasticity tensor D^{m}_{mnrs}, i.e.

$$U_{klmn}\,(\boldsymbol{r}-\boldsymbol{r}') = -\frac{1}{2}\left(\frac{\partial^2 G_{km}\,(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_l \partial x_n} + \frac{\partial^2 G_{lm}\,(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_k \partial x_n}\right), \tag{2.18}$$

and the volume integration in equation (2.17) extends over all the periodic cells of the composite material, i.e. over the entire composite. The Green's function tensor is defined by the Fourier integral (Barnett & Swanger 1971; Barnett 1972; Mura 1987; Walker 1993),

$$G_{ij}\,(\boldsymbol{r}-\boldsymbol{r}') = \int_{-\infty}^{\infty}\!\!\iint \frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\,\frac{M^{-1}_{ij}\,(\boldsymbol{\zeta})}{K^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot(\boldsymbol{r}-\boldsymbol{r}')}, \tag{2.19}$$

in which the vector $\boldsymbol{\zeta}$ is now defined by the relation $\zeta_i = K_i/K$ with $K = \sqrt{(K_q K_q)}$ denoting the magnitude of the vector $\boldsymbol{K} = (K_1, K_2, K_3)$.

Due to the singularity of the function $U_{klmn}(\boldsymbol{r} - \boldsymbol{r}')$, the volume integral in equation (2.17) does not exist in the Riemann sense. This difficulty is easily avoided by noting that in the derivation the actual integral equation takes on the form

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} + \frac{1}{2}\left\{\frac{\partial}{\partial x_l}\iiint_V \frac{\partial G_{km}(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_n}\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}')\right.$$
$$\left. + \frac{\partial}{\partial x_k}\iiint_V \frac{\partial G_{lm}(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_n}\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}')\right\}. \tag{2.20}$$

When the Green's function is expressed as a Fourier integral, the differentiation with respect to the variables x_l and x_k in (2.20) may be taken under the integral signs, and therefore equation (2.17) is correct when $U_{klmn}(\boldsymbol{r} - \boldsymbol{r}')$ is represented as the derivative of the Fourier integral (cf. Appendix G of Walker *et al.* 1989). However, when the Green's function is written as an algebraic expression, we must either use equation (2.20), or else use (2.17) in the Cauchy principal value sense, and observe that the convective differentiation of the volume integral by means of Leibniz's rule produces an additional jump term as noted by Bui (1978).

In Walker *et al.* (1989, 1990) we showed that the Poisson sum formula could be applied to the Fourier series representation of the polarization strain increment, $\Delta p_{kl}(\boldsymbol{r})$, to turn it into an equivalent Green's function formulation. The Poisson sum formula transforms the unprimed Fourier summation in equation (2.11) into an implied summation over all the unit cells in the periodic lattice, so that in the Green's function summation, equation (2.11) is replaced by the equivalent sum,

$$\Delta p_{kl}(\boldsymbol{r}) = -\iiint_V U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'). \tag{2.21}$$

Thus equation (2.17) does not satisfy the requirement that $\langle\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r})\rangle = \Delta\varepsilon^{0}_{kl}$, since equation (2.21) corresponds to the unprimed or full Fourier series summation. In order to effectively prime the formulation in equation (2.17) we must subtract the zeroth or volume averaged term, equation (2.13), and write

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} + \Delta p_{kl}(\boldsymbol{r}) - \langle\Delta p_{kl}(\boldsymbol{r})\rangle$$
$$= \Delta\varepsilon^{0}_{kl} - \iiint_V U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}')$$
$$+ \frac{1}{V_{\mathrm{c}}}\iiint_{V_{\mathrm{c}}}\mathrm{d}V(\boldsymbol{r})\iiint_V U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'). \tag{2.22}$$

3. Remarks on volume averaging

The volume integration in equation (2.17) represents that contribution to the total strain increment arising from fictitious body force stress increments,

$\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')$, induced in the cylindrical fibres embedded in the matrix constituent phase of the composite material. In a numerical evaluation of this integral only the contribution from a finite number of cylinders, surrounding the field point $\boldsymbol{r}$ and enclosed within the volume V_s with bounding surface S, are considered. Beyond S the material corresponds to the matrix constituent phase of the composite, as depicted in figure 1 *a*.

If we now suppose that the outer region beyond S, which extends to infinity, is comprised of a material which has the volume averaged elastic properties of a unit cell, as in figure 1 *b*, then the average strain increment on the bounding surface S induced by an overall uniform stress increment $\Delta\sigma^0_{kl}$ applied at infinity is given by

$$\Delta\varepsilon^0_{ij} = (D^0_{ijkl})^{-1}\Delta\sigma^0_{kl}, \tag{3.1}$$

where D^0_{ijkl} is the elasticity tensor of the homogenized (volume averaged) outer phase. The incremental surface traction on S in figure 1 *a* corresponding to this strain increment would be

$$\Delta t^0_i(\boldsymbol{r}') = n_j(\boldsymbol{r}')\,D^{\mathrm{m}}_{ijkl}\Delta\varepsilon^0_{kl}. \tag{3.2}$$

However, since the material in the outer phase corresponds to the matrix constituent phase, the average strain increment on S (when equation (2.17) – corresponding to figure 1 *a* – is the governing integral equation) will be given by

$$\Delta\varepsilon^1_{ij} = (D^{\mathrm{m}}_{ijkl})^{-1}\Delta\sigma^0_{kl}, \tag{3.3}$$

and the corresponding incremental surface traction on S in figure 1 *a* is

$$\Delta t^1_i(\boldsymbol{r}') = n_j(\boldsymbol{r}')\,D^{\mathrm{m}}_{ijkl}\Delta\varepsilon^1_{kl} = n_j(\boldsymbol{r}')\,\Delta\sigma^0_{ij}. \tag{3.4}$$

Hence, the total strain increment at any point inside the surface S in figure 1 *a* will be identical to that in figure 1 *b* if we add an incremental layer of depolarizing surface traction on the surface S in figure 1 *a* of magnitude

$$\Delta t_i(\boldsymbol{r}') = \Delta t^0_i(\boldsymbol{r}') - \Delta t^1_i(\boldsymbol{r}') = n_j(\boldsymbol{r}')\,D^{\mathrm{m}}_{ijkl}(\Delta\varepsilon^0_{kl} - \Delta\varepsilon^1_{kl}). \tag{3.5}$$

It is clear on physical grounds that the strain increment at any point inside S, when only a finite number of cylinders contribute to the result, is approximately modelled by figure 1 *b* rather than by figure 1 *a*. That is, equation (2.22) which gives $\langle\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r})\rangle = \Delta\varepsilon^0_{kl}$ in any unit cell in the composite, either inside or outside of S, is the appropriate equation to determine the local variation of the total strain increment.

Thus, if we make the situations depicted in figure 1 *a* and *b* equivalent to each other by adding a layer of incremental depolarizing surface traction of amount given by equation (3.5) on the surface S in figure 1 *c*, then the total strain increment will be given by the volume sources in equation (2.17) plus the strain increment arising from the layer of surface traction on S, namely,

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^0_{kl} - \iiint_{V_S} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}') + \iint_S \Gamma_{klm}(\boldsymbol{r}-\boldsymbol{r}')\,\Delta t_m(\boldsymbol{r}')\,\mathrm{d}S(\boldsymbol{r}'), \tag{3.6}$$

with $\Delta t_m(\boldsymbol{r}')$ given by (3.5), and where the third rank tensor $\Gamma_{klm}(\boldsymbol{r}-\boldsymbol{r}')$ gives the kl component of the total strain increment at point $\boldsymbol{r}$ due to the mth component of a surface traction increment at point $\boldsymbol{r}'$ in the infinite matrix with elasticity tensor D^{m}_{mnrs}, i.e.

$$\Gamma_{klm}(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{2}\left(\frac{\partial G_{km}(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_l} + \frac{\partial G_{lm}(\boldsymbol{r}-\boldsymbol{r}')}{\partial x_k}\right). \tag{3.7}$$

We now show that the additional resultant strain increment at $\boldsymbol{r}$ due to this incremental surface traction on S in the surface integral of equation (3.6) is equal to minus the volume average of the polarization strain increment in equation (2.22), namely, $-\langle\Delta p_{kl}(\boldsymbol{r})\rangle$. This is physically evident by noting that the distribution of the depolarizing layer of surface traction on S in figure 1*a* is equivalent to replacing the material in the outer phase with the homogenized material, which corresponds to determining the total strain increment from equation (2.22).

Let the field point $\boldsymbol{r}$ be situated near the centre of the volume V_s in figure 1*b*, and consider the volume integration in equation (2.22) to extend only over the finite number of unit cells enclosed within S. The total strain increment at $\boldsymbol{r}$ is then given by

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} - \iiint_{V_s} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}')$$

$$+\frac{1}{V_{\mathrm{c}}}\iiint_{V_{\mathrm{c}}}\mathrm{d}V(\boldsymbol{r})\iiint_{V_s} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'), \tag{3.8}$$

where V_{c} is the volume of the unit cell containing the field point $\boldsymbol{r}$. Now at any source point $\boldsymbol{r}'$, the incremental form of Hooke's law is

$$\Delta\sigma_{kl}(\boldsymbol{r}') = D_{klrs}(\boldsymbol{r})\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}') = \left(D^{\mathrm{m}}_{klrs} + \delta D_{klrs}(\boldsymbol{r}')\right)\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'), \tag{3.9}$$

so that

$$\delta D_{klrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}') = \Delta\sigma_{kl}(\boldsymbol{r}') - D^{\mathrm{m}}_{klrs}\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'). \tag{3.10}$$

The last volume averaged term in equation (3.8) can then be written as

$$\langle\Delta p_{kl}(\boldsymbol{r})\rangle =$$

$$-\frac{1}{V_{\mathrm{c}}}\iiint_{V_{\mathrm{c}}}\mathrm{d}V(\boldsymbol{r})\iiint_{V_s} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')\left(\Delta\sigma_{mn}(\boldsymbol{r}') - D^{\mathrm{m}}_{mnrs}\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\right)\mathrm{d}V(\boldsymbol{r}'). \tag{3.11}$$

By the mean value theorem the averaging with respect to $\boldsymbol{r}$ is equivalent to writing (3.11) in the form

$$\langle\Delta p_{kl}(\boldsymbol{r})\rangle = -\iiint_{V_s} U_{klmn}(\boldsymbol{\varrho}-\boldsymbol{r}')\left(\Delta\sigma_{mn}(\boldsymbol{r}') - D^{\mathrm{m}}_{mnrs}\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\right)\mathrm{d}V(\boldsymbol{r}'), \tag{3.12}$$

where $\boldsymbol{\varrho}$ is the position vector to some effective mean point contained in the volume V_{c}. Since the total strain increment determined by (3.8) satisfies the volume average condition, it is clear that the source terms in (3.12) fluctuate

as much above their average counterparts as below them, so that we may write (3.12) in the form

$$\langle \Delta p_{kl}(\boldsymbol{r}) \rangle = -\iiint_{V_S} U_{klmn}(\boldsymbol{\varrho} - \boldsymbol{r}') \left(\Delta\sigma^0_{mn} - D^{\mathrm{m}}_{mnrs} \Delta\varepsilon^0_{rs} \right) \mathrm{d}V(\boldsymbol{r}'). \tag{3.13}$$

The volume integral can now be transformed, by means of Gauss's divergence theorem and the relation $\partial G_{km}(\boldsymbol{r} - \boldsymbol{r}')/\partial x_n = -\partial G_{km}(\boldsymbol{r} - \boldsymbol{r}')/\partial x'_n$, into a surface integral over S, so that

$$\langle \Delta p_{kl}(\boldsymbol{r}) \rangle = -\iint_S \Gamma_{klp}(\boldsymbol{\varrho} - \boldsymbol{r}') \, \Delta t_p(\boldsymbol{r}') \, \mathrm{d}S(\boldsymbol{r}'), \tag{3.14}$$

where, with the use of equation (3.3),

$$\Delta t_p(\boldsymbol{r}') = n_q(\boldsymbol{r}') \left[D^{\mathrm{m}}_{pqrs} \Delta\varepsilon^0_{rs} - \Delta\sigma^0_{pq} \right] = n_q(\boldsymbol{r}') \, D^{\mathrm{m}}_{pqrs} \left(\Delta\varepsilon^0_{rs} - \Delta\varepsilon^1_{rs} \right), \tag{3.15}$$

which shows that the application of the depolarizing surface traction increment in equation (3.6) is equivalent to subtracting the volume average of the polarization strain increment in equation (2.22).

The integral in equation (2.17), when applied to an infinite lattice, is conditionally convergent. That is, it depends on the shape of the infinite outer surface S in figure 1*b*. This arises from the fact that the two-dimensional Green's function derivative, $\Gamma_{klm}(\boldsymbol{r} - \boldsymbol{r}')$, decreases as r^{-1} $(= |\boldsymbol{r} - \boldsymbol{r}'|^{-1})$ as we recede from the field point. However, if the infinite outer surface is cylindrical, the incremental surface area per unit length is $\mathrm{d}S = r\,\mathrm{d}\theta$, so that $\Gamma_{klm}(\boldsymbol{r} - \boldsymbol{r}')\,\mathrm{d}S(\boldsymbol{r}')$ does not decrease with increasing r. *Even when the surface S is infinitely removed and we sum over an infinite number of unit cells, the effect of the incremental surface traction layer on S contributes a finite amount, $-\langle \Delta p_{kl}(\boldsymbol{r}) \rangle$, to the total strain increment.*

We may now examine the convergence of the volume integrals in the integral equation (2.22). By the mean value theorem the integral equation may be written as

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^0_{kl} + \iiint_V \left(U_{klmn}(\boldsymbol{\varrho} - \boldsymbol{r}') - U_{klmn}(\boldsymbol{r} - \boldsymbol{r}') \right) \delta D_{mnrs}(\boldsymbol{r}') \, \Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}') \, \mathrm{d}V(\boldsymbol{r}'). \tag{3.16}$$

If we examine the contribution to the integral from a thin annular slice when $|\boldsymbol{r}'| \gg (|\boldsymbol{\varrho}|$ or $|\boldsymbol{r}|)$, we find, on expanding the two-dimensional $\boldsymbol{U}$ tensors into Taylor series and retaining only the first term, that the integrand per unit length can be written approximately as

$$(x_i - \varrho_i) \frac{\partial U_{klmn}(\boldsymbol{r}')}{\partial x'_i} \delta D_{mnrs}(\boldsymbol{r}') \, \Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}') \, r' \, \mathrm{d}r' \, \mathrm{d}\theta. \tag{3.17}$$

Now the derivative of the $\boldsymbol{U}$ tensor decreases as $(r')^{-3}$, so that the integrand decreases as $(r')^{-2}$ and the contribution from the volume in the region extending from r' to ∞ decreases to zero as the radius r' increases. The volume integral in (2.22) is therefore absolutely convergent and is now independent of the shape of

S when the volume integration extends over an infinite number of unit cells. This independence of $\Delta\varepsilon_{kl}^{\mathrm{T}}(\boldsymbol{r})$ on the shape of the surface S is also clear from physical principles, since in figure 1 *b* the actual fluctuating incremental surface traction on S—which would exist if the outer region beyond S were composed of unit cells—is equipollent to the average value corresponding to that of the homogenized outer medium. Only the difference between the fluctuating and average incremental surface traction on S in figure 1 *b* contributes to the total strain increment at the field point $\boldsymbol{r}$, and since the true varying incremental surface traction varies as much above the mean value as below it, this contribution vanishes at the field point in accordance with Saint-Venant's hypothesis.

The increase in convergence of the integrand, as evidenced in equations (3.16) and (3.17), has been achieved by replacing the conditionally convergent integral in equation (2.17) with the absolutely convergent Frullani difference integral (Jeffreys & Jeffreys 1988) in equation (2.22). Since the volume averaged polarization term in (2.22) is constant, the subtraction of the volume averaged term essentially represents a *renormalization* of the applied strain increment, $\Delta\varepsilon_{kl}^{0}$, due to embedding the surface S, which may be at 'infinity', in a homogenized medium 'beyond infinity'. In some instances, for example (Mura 1987) in a three-dimensional periodic distribution of spherical inclusions with dilatational eigenstrains, the omission of the volume averaged term, $\langle\Delta p_{kl}(\boldsymbol{r})\rangle$, from the integral equation will result in the polarization term, $\Delta p_{kl}(\boldsymbol{r})$, becoming an infinite diverging volume integral. The *renormalization* brought about by subtracting the volume averaged polarization essentially subtracts two integrals, each of which may be conditionally convergent or even divergent, to produce a convergent Frullani integral equation.

The conditional convergence (or divergence) of the volume integral in (2.17) has a curious history. By effectively subtracting the average polarization from (2.17) to obtain the absolutely convergent polarization difference in equation (2.22), Clausius and Mosotti, in an analogous electrical problem, correctly computed (Jackson 1975) the effective dielectric constant of a material in which the atoms were supposed to be conducting spheres embedded in a matrix dielectric material. The same technique is used in optics (Born & Wolf 1980) to obtain the Lorentz–Lorenz formula which relates the refractive index of a material to its density. In 1892 Lord Rayleigh treated the question of determining the effective thermal/electrical conductivity (Rayleigh 1892) of spheres and cylinders arranged in rectangular order in a matrix of uniform conductivity. Since he did not subtract the volume averaged term in an explicit manner, the series which arose in his calculations showed conditional convergence, in which the electric field or temperature in the unit cell depended on the shape of the surface S separating the finite number of spheres (cylinders) from the infinite outer matrix beyond S. By assuming that the shape of S corresponded to a rectangular shaped needle orientated in the direction of the electric field or temperature gradient, the effective volume average of the polarization term vanishes as the needle becomes longer, and Rayleigh obtained the correct value for the conductivity. In taking S to be needle shaped he was emmulating a procedure used by Lord Kelvin to determine the effective magnetic field inside a magnetic material forty one years earlier. Kelvin, and Poisson before him, had noted (Whittaker 1989) that the magnetic intensity inside a small cavity within the medium depended on the shape of the cavity. Kelvin showed that the actual magnetic field within the medium was represented by the value of the field within a narrow tubular cavity tangential to the

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direction of magnetization. In a letter to Stokes in 1849, Kelvin (1990) called the needle shaped cavity S a *split* in the magnetic medium.

The procedure used by Kelvin and Rayleigh, when cast into the present context, amounts to writing the total strain increment as

$$\Delta\varepsilon_{kl}^{\mathrm{T}}(\boldsymbol{r}) = \Delta\varepsilon_{kl}^{0} + \Delta p_{kl}(\boldsymbol{r}, S) - \langle \Delta p_{kl}(\boldsymbol{r}, S)\rangle, \tag{3.18}$$

where the explicit dependence of the polarization and its volume average on the surface of separation S is noted. When S is needle shaped in the direction of the applied field, the polarization average can be transformed to a surface integral over the needle's surface, S. If the integrand has no component perpendicular to the needle, as will be the case when the needle is parallel to the applied field, and the needle is highly elongated, the surface integral and hence the volume averaged polarization tends to zero. Rayleigh was aware that the omission of $\langle \Delta p_{kl}(\boldsymbol{r}, S)\rangle$ from (3.18) is allowed, and is compensated by the corresponding change in the sum for $\Delta p_{kl}(\boldsymbol{r}, S)$ when S is an infinite rectangular needle, for in a footnote he states: 'It would be otherwise if the infinite rectangle were supposed to be of another shape, e.g. to be square.' This technique of omitting the volume averaged term and summing over a long needle shaped region gives the correct result for electrical and temperature problems, but in the elastic case the total strain increment depends on all the components of the overall applied stress or strain increment, and therefore the sum would have to be taken over different needle orientations for the different components.

Doubts were cast on Rayleigh's procedure (but not his results) by Levine (1966) and Jeffrey (1973). Later, a physical basis for the conditionally convergent sums was explained in the electrical context by McPhedran & McKenzie (1978) and McKenzie *et al.* (1978), in which they showed that the shape dependence arose from the depolarizing field on the surface S.

To avoid the conditionally convergent sums, Jeffrey (1973) used a general mathematical technique devised by Batchelor (1972) in computing the effective conductivity of a random suspension of spheres, while McCoy & Beran (1976) essentially used the volume averaging technique to achieve absolute convergence.

Furuhashi *et al.* (1981) and Mura (1987) have commented on the conditional convergence (and divergence) of series which arise in computing the effect of periodic eigenstrains via a Green's function approach. They observe that by subtracting a constant comparison series, in which the field variable is evaluated at the centre of each unit cell, the series difference is absolutely convergent. They note that an arbitrary constant can be added to the periodic sum and determine this constant by requiring the volume averaged stress to satisfy the condition of periodicity. This condition is met by requiring the volume averaged strain to equal the applied strain, which, for the periodic eigenstrain problem, is zero. However, the series difference plus the constant arising from periodicity does not satisfy the volume average condition. They then proceed to rectify this deficiency by subtracting a further volume averaged term. It is clear that the comparison series can be removed from their volume integration since it is independent of the integration variable, and the original comparison series then cancels the integrated comparison series. Since the comparison series vanishes from the end result, it is evident that the subtraction of the original comparison series is not required. The absolute convergence of the series is effected by the subtraction of

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the volume averaged term alone; a result in harmony with the contention of this paper.

The latest occurrence of these conditionally convergent sums appears in the paper by Horii & Sahasakmontri (1990) in which they calculate the stress distribution and stress intensity factors in a doubly periodic array of cracks. They give the history of the crack problem as it emerged during the last two decades, and its resolution by averaging techniques.

4. Numerical solution of the integral equation

In the remainder of the paper we shall be concerned with the two-dimensional form of the integral equation (2.22) as applied to a periodic lattice.

We can make the implied summation in equation (2.22) explicit by introducing the matrix/vector relations

$$\boldsymbol{m} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{L} = (L_1, L_2)\,, \tag{4.1}$$

where m_1, m_2 are integers. Within a given unit cell, let $\boldsymbol{r} = (x_1, x_2)$ be the position vector to the field point with respect to a cartesian coordinate system whose origin is located at the centre of the unit cell, and let $\boldsymbol{r}' = (x_1', x_2')$ be the position vector to the corresponding source point in the same unit cell which contains the field point $\boldsymbol{r}$. The corresponding position vector to a similar source point in a unit cell located m_1 and m_2 periodic lengths upstream in the x_1 and x_2 directions is given by

$$\boldsymbol{r}'' = (x_1' + m_1 L_1)\,\boldsymbol{i} + (x_2' + m_2 L_2)\,\boldsymbol{j} = \boldsymbol{r}' + \boldsymbol{m}\cdot\boldsymbol{L}. \tag{4.2}$$

Equation (2.22) can then be written in the two-dimensional form

$$\begin{aligned}\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} &- \sum_{m_1=0}^{\pm\infty}\sum_{m_2=0}^{\pm\infty} \iint_{A_{\mathrm{c}}(m_1,m_2)} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}'')\,\delta D_{mnrs}(\boldsymbol{r}'')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'')\,\mathrm{d}S(\boldsymbol{r}'') \\ &+ \frac{1}{A_{\mathrm{c}}} \sum_{m_1=0}^{\pm\infty}\sum_{m_2=0}^{\pm\infty} \iint_{A_{\mathrm{c}}(0,0)} \mathrm{d}S(\boldsymbol{r}) \\ &\times \iint_{A_{\mathrm{c}}(m_1,m_2)} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}'')\,\delta D_{mnrs}(\boldsymbol{r}'')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'')\,\mathrm{d}S(\boldsymbol{r}'')\,,\end{aligned} \tag{4.3}$$

where $A_{\mathrm{c}}(m_1, m_2)$ denotes the area of the unit cell whose centre is located at the point $(m_1 L_1, m_2 L_2)$, and $A_{\mathrm{c}}(0,0)$ is the area of the unit cell containing the field point $\boldsymbol{r}$. For brevity we will set $A_{\mathrm{c}} = A_{\mathrm{c}}(0,0)$.

From the periodicity of the lattice it follows that

$$\begin{Bmatrix} \delta D_{mnrs}(\boldsymbol{r}'') \\ \Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'') \\ \mathrm{d}S(\boldsymbol{r}'') \end{Bmatrix} = \begin{Bmatrix} \delta D_{mnrs}(\boldsymbol{r}'+\boldsymbol{m}\cdot\boldsymbol{L}) \\ \Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}'+\boldsymbol{m}\cdot\boldsymbol{L}) \\ \mathrm{d}S(\boldsymbol{r}'+\boldsymbol{m}\cdot\boldsymbol{L}) \end{Bmatrix} = \begin{Bmatrix} \delta D_{mnrs}(\boldsymbol{r}') \\ \Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}') \\ \mathrm{d}S(\boldsymbol{r}') \end{Bmatrix}, \tag{4.4}$$

so that the integral equation can be written as

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} - \sum_{m_1=0}^{\pm\infty}\sum_{m_2=0}^{\pm\infty}\iint_{A_c} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{m}\cdot\boldsymbol{L})\,\delta D_{mnrs}(\boldsymbol{r}')\,\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}S(\boldsymbol{r}') + \frac{1}{A_c}\sum_{m_1=0}^{\pm\infty}\sum_{m_2=0}^{\pm\infty}\iint_{A_c}\mathrm{d}S(\boldsymbol{r}) \times \iint_{A_c} U_{klmn}(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{m}\cdot\boldsymbol{L})\,\delta D_{mnrs}(\boldsymbol{r}')\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}S(\boldsymbol{r}'), \tag{4.5}$$

where the integration with respect to the source point $\boldsymbol{r}'$ now extends only over the unit cell containing the field point $\boldsymbol{r}$. The contribution from the source points in the remainder of the lattice is now assimilated to the summation over m_1 and m_2.

In a numerical evaluation of equation (4.5), we denote the double summation over a finite number of terms from $m_1 = 0$ to $\pm n$ and $m_2 = 0$ to $\pm n$ by $S_{kl}(\boldsymbol{r}, n)$. It is apparent that the zeroth approximation,

$$\Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r}) = \Delta\varepsilon^{0}_{kl} + S_{kl}(\boldsymbol{r}, 0), \tag{4.6}$$

in figure 2*a* corresponds to the generalized self-consistent method in which the fibre is surrounded by the constituent matrix phase in a single unit cell which is itself embedded in a homogenized outer medium. Carrying out the double summation to $n = 1$ and $n = 2$ shows that the approximations $S_{kl}(\boldsymbol{r}, 1)$ and $S_{kl}(\boldsymbol{r}, 2)$ correspond to considering the interaction of the unit cell with its eight nearest neighbours in figure 2*b* and the additional sixteen adjacent outer neighbours in figure 2*c*, when these are embedded in the homogenized outer medium. The successive approximations converge very rapidly with increasing n, and by comparing the results for $n = 0$ with those for $n = 1, 2, \ldots,$ etc. we can estimate the accuracy of the generalized self-consistent method in comparison with the exact solution for an infinite periodic lattice. The rapid convergence of the successive Poisson summations, where effective convergence is obtained with ± 1 terms in each direction, should be contrasted with the much slower convergence of the Fourier series summation, which requires at least ± 50 terms in each direction to achieve comparable accuracy. This observation is in keeping with the well-known fact (Wallace 1984) that if a Fourier series is slowly convergent, then its Poisson sum counterpart converges rapidly, and vice versa.

A numerical solution to the integral equation can be achieved by dividing the unit cell into a number of discrete subvolumes, and by approximating the strain increment, $\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')$, in the βth subvolume integral with its average value in the subvolume, namely, by

$$\Delta\varepsilon^{\mathrm{T}\beta}_{rs} = \frac{1}{A_\beta}\iint_{A_\beta}\Delta\varepsilon^{\mathrm{T}}_{rs}(\boldsymbol{r}')\,\mathrm{d}S(\boldsymbol{r}'), \tag{4.7}$$

where A_β is the cross-sectional area of the subvolume.

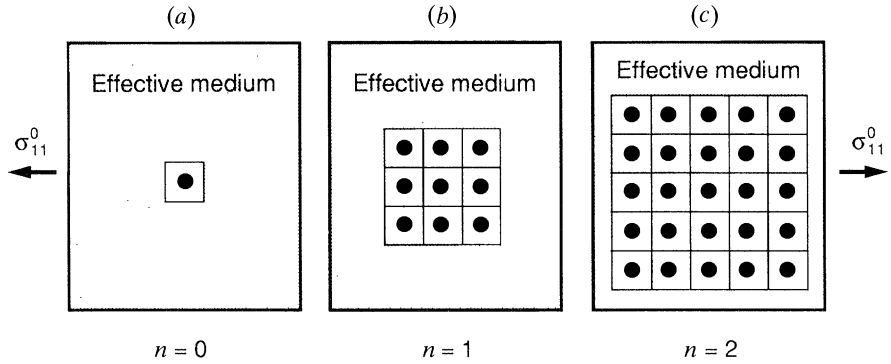

Figure 2. (*a*) Generalized self-consistent method ($n = 0$); (*b*) with 8 nearest neighbours included ($n = 1$); (*c*) with 24 nearest neighbours included ($n = 2$).

Let there be N subvolumes in the unit cell, with M subvolumes in the fibre and $N - M$ subvolumes in the matrix. It may be noted that if any function $g(\boldsymbol{r})$ is approximated as being piecewise constant and equal to g^β in the βth subvolume, then

$$\langle g(\boldsymbol{r})\rangle = \frac{1}{A_c}\iint_{A_c} g(\boldsymbol{r})\,\mathrm{d}S\,(\boldsymbol{r}) = \sum_{\beta=1}^{N} f^\beta g^\beta, \tag{4.8}$$

where $f^\beta = A_\beta/A_c$ is the volume fraction of the subvolume β such that $\sum_{\beta=1}^{N} f^\beta = 1$.

We now suppose that the field point $\boldsymbol{r}$ is located in the αth subvolume, and we locate the origin of the local coordinate system ξ_1, ξ_2 at its centroid, (x_1^α, x_2^α), so that $\boldsymbol{r} = (x_1^\alpha + \xi_1, x_2^\alpha + \xi_2)$. Local coordinate systems η_1, η_2 are also located at the centroid of each subvolume, and when the source point is located in the βth subvolume we write $\boldsymbol{r}' = (x_1^\beta + \eta_1, x_2^\beta + \eta_2)$. Then, following the notation of equation (2.15), the integral equation may be written in the form

$$\Delta\varepsilon_{kl}^{\mathrm{T}}(x_1^\alpha + \xi_1, x_2^\alpha + \xi_2) = \Delta\varepsilon_{kl}^0 + \sum_{\beta=1}^{N}\Delta p_{kl}^{\alpha\beta} - \left\langle \sum_{\beta=1}^{N}\Delta p_{kl}^{\alpha\beta}\right\rangle, \tag{4.9}$$

where, for rectangular subvolumes,

$$\Delta p_{kl}^{\alpha\beta} = -\sum_{m_1=0}^{\pm n}\sum_{m_2=0}^{\pm n}\int_{\eta_1=-L_1^\beta/2}^{L_1^\beta/2}\mathrm{d}\eta_1\int_{\eta_2=-L_2^\beta/2}^{L_2^\beta/2}\mathrm{d}\eta_2$$

$$\times U_{klmn}\left(x_1^\alpha - x_1^\beta + \xi_1 - \eta_1 - m_1L_1,\ x_2^\alpha - x_2^\beta + \xi_2 - \eta_2 - m_2L_2\right)\delta D_{mnrs}^\beta\Delta\varepsilon_{rs}^{\mathrm{T}\beta}, \tag{4.10}$$

in which L_1^β, L_2^β are the lengths of the sides of the βth rectangular subvolume in the x_1, x_2 directions, and where $\delta D_{mnrs}^\beta = D_{mnrs}^{\mathrm{f}} - D_{mnrs}^{\mathrm{m}}$ or 0 according as the subvolume β is in the fibre or matrix, respectively.

We may now volume average the integral equation in (4.9) over the αth sub-

volume using equation (4.7) to obtain

$$\Delta\varepsilon_{kl}^{\mathrm{T}\alpha} = \Delta\varepsilon_{kl}^{0} - \sum_{\beta=1}^{M} f^{\beta} S_{klrs}^{\alpha\beta} \Delta\varepsilon_{rs}^{\mathrm{T}\beta} + \sum_{\gamma=1}^{N} \sum_{\beta=1}^{M} f^{\gamma} f^{\beta} S_{klrs}^{\gamma\beta} \Delta\varepsilon_{rs}^{\mathrm{T}\beta}, \tag{4.11}$$

where

$$f^{\beta} S_{klrs}^{\alpha\beta} = \frac{1}{L_1^{\alpha} L_2^{\alpha}} \sum_{m_1=0}^{\pm n} \sum_{m_2=0}^{\pm n} \int_{\xi_1=-L_1^{\alpha}/2}^{L_1^{\alpha}/2} \mathrm{d}\xi_1 \int_{\xi_2=-L_2^{\alpha}/2}^{L_2^{\alpha}/2} \mathrm{d}\xi_2 \int_{\eta_1=-L_1^{\beta}/2}^{L_1^{\beta}/2} \mathrm{d}\eta_1 \int_{\eta_2=-L_2^{\beta}/2}^{L_2^{\beta}/2} \mathrm{d}\eta_2$$
$$\times U_{klmn}\left(x_1^{\alpha} - x_1^{\beta} + \xi_1 - \eta_1 - m_1 L_1,\ x_2^{\alpha} - x_2^{\beta} + \xi_2 - \eta_2 - m_2 L_2\right) \delta D_{mnrs}^{\beta}, \tag{4.12}$$

and the volume average of the polarization strain increment is evaluated using equation (4.8). The matrix tensor $S_{klrs}^{\alpha\beta}$ is akin to Eshelby's tensor (Eshelby 1957; Mura 1987) for an ellipsoidal inclusion, but accounts for the interaction between a rectangular subvolume with its neighbours in the infinite lattice. Its counterpart in the Fourier series subvolume representation of equation (2.5) is given by equations (22) and (24) of Walker *et al.* (1991, 1992), respectively. If the cross-section of the subvolume were of elliptical shape and the tensor δD_{mnrs}^{β} in equation (4.12) is replaced by D_{mnrs}^{m}, then the term $f^{\alpha} S_{klrs}^{\alpha\alpha}$ would correspond precisely to Eshelby's tensor for the αth subvolume.

It may be noted that the superscript β in equation (4.11) is summed only over the subvolumes in the fibre, where $(1 \leq \beta \leq M)$, since $\delta D_{mnrs}^{\beta} = 0$ if the βth subvolume resides in the matrix, so that

$$S_{klrs}^{\alpha\beta} = 0 \quad \text{for} \quad M < \beta \leq N. \tag{4.13}$$

Thus only M unknowns (associated with the subvolumes in the fibre) are involved in equation (4.11). When this relation is assembled columnwise for each subvolume α, the solution to the $6M \times 6M$ system of equations can be obtained by LU decomposition (Press *et al.* 1986). We then have to solve the system of equations

$$\sum_{\beta=1}^{M} B_{klrs}^{\alpha\beta} \Delta\varepsilon_{rs}^{\mathrm{T}\beta} = \Delta\varepsilon_{kl}^{0} \quad \text{for} \quad \alpha = 1, 2, \ldots, M, \tag{4.14}$$

where

$$B_{klrs}^{\alpha\beta} = \delta^{\alpha\beta} I_{klrs} + f^{\beta} S_{klrs}^{\alpha\beta} - \sum_{\gamma=1}^{N} f^{\gamma} f^{\beta} S_{klrs}^{\gamma\beta}, \tag{4.15}$$

with $I_{klrs} = \frac{1}{2}(\delta_{kr}\delta_{ls} + \delta_{ks}\delta_{lr})$ denoting the fourth rank identity tensor and $\delta^{\alpha\beta}$ the matrix Kronecker delta.

Equation (4.11) can then be used with the known values of $\Delta\varepsilon_{kl}^{\mathrm{T}\beta}$ in the fibre subvolumes, where $(1 \leq \beta \leq M)$, to compute the values of $\Delta\varepsilon_{kl}^{\mathrm{T}\alpha}$ in the matrix subvolumes, where $(M < \alpha \leq N)$. Once the total strain increment $\Delta\varepsilon_{kl}^{\mathrm{T}\beta}$ has been determined for all the subvolumes in the unit cell, further resolution can be acquired by using equations (4.9) and (4.10) to determine the total strain increment $\Delta\varepsilon_{kl}^{\mathrm{T}}(x_1^{\alpha} + \xi_1, x_2^{\alpha} + \xi_2)$ at any point (ξ_1, ξ_2) within the αth subvolume.

5. Matrix assembly of discretized integral equation

The solution of the integral equation in (4.14) requires the evaluation of the matrix tensor $S_{klrs}^{\alpha\beta}$ from equation (4.12). For an isotropic matrix constituent phase with Lamé constants, λ^{m} and μ^{m}, the Green's function (2.19) can be written as (Chen & Young 1977; Walker *et al.* 1989)

$$G_{km}(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{8\pi\mu^{\mathrm{m}}}\left(\delta_{km}\frac{\partial^2|\boldsymbol{r}-\boldsymbol{r}'|}{\partial x_q \partial x_q} - \frac{\lambda^{\mathrm{m}}+\mu^{\mathrm{m}}}{\lambda^{\mathrm{m}}+2\mu^{\mathrm{m}}}\frac{\partial^2|\boldsymbol{r}-\boldsymbol{r}'|}{\partial x_k \partial x_m}\right) \tag{5.1}$$

and differentiated to yield the two-dimensional form of the tensor $U_{klmn}(\boldsymbol{r})$ in equation (2.18) in the form

$$U_{klmn}(\boldsymbol{r}-\boldsymbol{r}') = -\frac{1}{16\pi\mu^{\mathrm{m}}}\int_{x_3'=-\infty}^{\infty} \mathrm{d}x_3' \left(\delta_{km}\frac{\partial^4|\boldsymbol{r}-\boldsymbol{r}'|}{\partial x_l \partial x_n \partial x_q \partial x_q} + \delta_{lm}\frac{\partial^4|\boldsymbol{r}-\boldsymbol{r}'|}{\partial x_k \partial x_n \partial x_q \partial x_q} - \frac{2(\lambda^{\mathrm{m}}+\mu^{\mathrm{m}})}{\lambda^{\mathrm{m}}+2\mu^{\mathrm{m}}}\frac{\partial^4|\boldsymbol{r}-\boldsymbol{r}'|}{\partial x_k \partial x_l \partial x_m \partial x_n}\right). \tag{5.2}$$

Alternatively, we can write the inverse Christoffel stiffness tensor for an isotropic matrix material in the form (Walker *et al.* 1989)

$$M_{ij}^{-1}(\boldsymbol{\zeta}) = \frac{\delta_{ij}}{\mu^{\mathrm{m}}} - \frac{\lambda^{\mathrm{m}}+\mu^{\mathrm{m}}}{\mu^{\mathrm{m}}(\lambda^{\mathrm{m}}+2\mu^{\mathrm{m}})}\zeta_i\zeta_j, \tag{5.3}$$

and when this relation is inserted into (2.19) and the required differentiation in (2.18) is carried out, the two-dimensional Fourier integral representation of $U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')$ can be written as

$$U_{klmn}(\boldsymbol{r}-\boldsymbol{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}^3\boldsymbol{K}}{(2\pi)^3}\left(\frac{\delta_{km}\zeta_l\zeta_n+\delta_{lm}\zeta_k\zeta_n}{2\mu^{\mathrm{m}}} - \frac{\lambda^{\mathrm{m}}+\mu^{\mathrm{m}}}{\mu^{\mathrm{m}}(\lambda^{\mathrm{m}}+2\mu^{\mathrm{m}})}\zeta_k\zeta_l\zeta_m\zeta_n\right) \times \mathrm{e}^{-\mathrm{i}\{K_1(x_1-x_1')+K_2(x_2-x_2')+K_3x_3\}}\int_{x_3'=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}K_3x_3'}\,\mathrm{d}x_3'. \tag{5.4}$$

The integration with respect to x_3' produces the term $2\pi\delta(K_3)$, where $\delta(K_3)$ is the Dirac delta function, and the subsequent integration with respect to K_3 then allows the two-dimensional form of $U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')$ to be written as

$$U_{klmn}(\boldsymbol{r}-\boldsymbol{r}') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}K_1\,\mathrm{d}K_2}{(2\pi)^2}\left(\frac{\delta_{km}\zeta_l\zeta_n+\delta_{lm}\zeta_k\zeta_n}{2\mu^{\mathrm{m}}} - \frac{\lambda^{\mathrm{m}}+\mu^{\mathrm{m}}}{\mu^{\mathrm{m}}(\lambda^{\mathrm{m}}+2\mu^{\mathrm{m}})}\zeta_k\zeta_l\zeta_m\zeta_n\right) \times \mathrm{e}^{-\mathrm{i}\{K_1(x_1-x_1')+K_2(x_2-x_2')\}}, \tag{5.5}$$

in which $\zeta_m = K_m/\sqrt{(K_1^2+K_2^2)}$ for $m = 1, 2$.

If the Green's function is given in algebraic form, the integrations involving $U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')$ must be carried out using equation (2.20). However, equation (2.17) may be used directly when the Fourier integral representation for $U_{klmn}(\boldsymbol{r}-\boldsymbol{r}')$ in equation (5.5) is used. When (5.5) is substituted into (4.12) and the integration over the interacting subvolumes α and β is effected, we obtain the

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following non-zero components of the matrix tensor $S^{\alpha\beta}_{klrs}$:

$$f^\beta S^{\alpha\beta}_{1111} = U^{\alpha\beta}_{1111}\delta D^\beta_{1111} + U^{\alpha\beta}_{1122}\delta D^\beta_{1122}, \tag{5.6}$$

$$f^\beta S^{\alpha\beta}_{1122} = U^{\alpha\beta}_{1111}\delta D^\beta_{1122} + U^{\alpha\beta}_{1122}\delta D^\beta_{1111}, \tag{5.7}$$

$$f^\beta S^{\alpha\beta}_{1133} = U^{\alpha\beta}_{1111}\delta D^\beta_{1122} + U^{\alpha\beta}_{1122}\delta D^\beta_{1122}, \tag{5.8}$$

$$f^\beta S^{\alpha\beta}_{2211} = U^{\alpha\beta}_{1122}\delta D^\beta_{1111} + U^{\alpha\beta}_{2222}\delta D^\beta_{1122}, \tag{5.9}$$

$$f^\beta S^{\alpha\beta}_{2222} = U^{\alpha\beta}_{1122}\delta D^\beta_{1122} + U^{\alpha\beta}_{2222}\delta D^\beta_{1111}, \tag{5.10}$$

$$f^\beta S^{\alpha\beta}_{2233} = U^{\alpha\beta}_{1122}\delta D^\beta_{1122} + U^{\alpha\beta}_{2222}\delta D^\beta_{1122}, \tag{5.11}$$

$$f^\beta S^{\alpha\beta}_{1212} = \left(U^{\alpha\beta}_{1212} + U^{\alpha\beta}_{1221}\right)\delta D^\beta_{1212}, \tag{5.12}$$

where

$$\delta D^\beta_{1111} = (\lambda^\beta - \lambda^{\mathrm{m}}) + 2(\mu^\beta - \mu^{\mathrm{m}}), \tag{5.13}$$

$$\delta D^\beta_{1122} = \lambda^\beta - \lambda^{\mathrm{m}}, \tag{5.14}$$

$$\delta D^\beta_{1212} = \mu^\beta - \mu^{\mathrm{m}}, \tag{5.15}$$

and

$$U^{\alpha\beta}_{1111} = -\frac{1}{2\mu^{\mathrm{m}}}\left(t^{\alpha\beta}_{1111} + t^{\alpha\beta}_{1122}\right) + \frac{\lambda^{\mathrm{m}} + \mu^{\mathrm{m}}}{2\mu^{\mathrm{m}}\left(\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}}\right)}\, t^{\alpha\beta}_{1111}, \tag{5.16}$$

$$U^{\alpha\beta}_{1122} = \frac{\lambda^{\mathrm{m}} + \mu^{\mathrm{m}}}{2\mu^{\mathrm{m}}\left(\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}}\right)}\, t^{\alpha\beta}_{1122}, \tag{5.17}$$

$$U^{\alpha\beta}_{2222} = -\frac{1}{2\mu^{\mathrm{m}}}\left(t^{\alpha\beta}_{2222} + t^{\alpha\beta}_{1122}\right) + \frac{\lambda^{\mathrm{m}} + \mu^{\mathrm{m}}}{2\mu^{\mathrm{m}}\left(\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}}\right)}\, t^{\alpha\beta}_{2222}, \tag{5.18}$$

$$U^{\alpha\beta}_{1212} = -\frac{1}{4\mu^{\mathrm{m}}}\left(t^{\alpha\beta}_{2222} + t^{\alpha\beta}_{1122}\right) + \frac{\lambda^{\mathrm{m}} + \mu^{\mathrm{m}}}{2\mu^{\mathrm{m}}\left(\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}}\right)}\, t^{\alpha\beta}_{1122}, \tag{5.19}$$

$$U^{\alpha\beta}_{1221} = -\frac{1}{4\mu^{\mathrm{m}}}\left(t^{\alpha\beta}_{1111} + t^{\alpha\beta}_{1122}\right) + \frac{\lambda^{\mathrm{m}} + \mu^{\mathrm{m}}}{2\mu^{\mathrm{m}}\left(\lambda^{\mathrm{m}} + 2\mu^{\mathrm{m}}\right)}\, t^{\alpha\beta}_{1122}. \tag{5.20}$$

In equations (5.13)–(5.15) the Lamé constants $(\lambda^\beta, \mu^\beta)$ are equal to $(\lambda^{\mathrm{f}}, \mu^{\mathrm{f}})$ or $(\lambda^{\mathrm{m}}, \mu^{\mathrm{m}})$ for isotropic constituent phases, depending on whether β represents a fibre or matrix subvolume. With the definitions

$$x_L = x_1^\alpha - x_1^\beta - \tfrac{1}{2}L_1^\alpha - m_1 L_1, \qquad x_U = x_1^\alpha - x_1^\beta + \tfrac{1}{2}L_1^\alpha - m_1 L_1, \tag{5.21}$$

and

$$y_L = x_2^\alpha - x_2^\beta - \tfrac{1}{2}L_2^\alpha - m_2 L_2, \qquad y_U = x_2^\alpha - x_2^\beta + \tfrac{1}{2}L_2^\alpha - m_2 L_2, \tag{5.22}$$

the components of the shape factor matrix tensor, $\boldsymbol{t}^{\alpha\beta}$, are defined by the relations

$$\begin{aligned} t^{\alpha\beta}_{1111} = \sum_{m_1=0}^{\pm n}\sum_{m_2=0}^{\pm n} \{ &- \Phi(x_L, x_U, y_L, y_U, L_1^\beta, L_2^\beta) + \Phi(x_L, x_U, y_L, y_U, L_1^\beta, -L_2^\beta) \\ &+ \Phi(x_L, x_U, y_L, y_U, -L_1^\beta, L_2^\beta) - \Phi(x_L, x_U, y_L, y_U, -L_1^\beta, -L_2^\beta) \\ &- \Psi(x_L, x_U, y_L, y_U, L_1^\beta, L_2^\beta)\}, \end{aligned} \tag{5.23}$$

$$t^{\alpha\beta}_{2222} = \sum_{m_1=0}^{\pm n}\sum_{m_2=0}^{\pm n}\{ -\Phi(y_L, y_U, x_L, x_U, L_1^\beta, L_2^\beta) + \Phi(y_L, y_U, x_L, x_U, L_1^\beta, -L_2^\beta) + \Phi(y_L, y_U, x_L, x_U, -L_1^\beta, L_2^\beta) - \Phi(y_L, y_U, x_L, x_U, -L_1^\beta, -L_2^\beta) - \Psi(y_L, y_U, x_L, x_U, L_1^\beta, L_2^\beta)\}, \tag{5.24}$$

$$t^{\alpha\beta}_{1122} = \sum_{m_1=0}^{\pm n}\sum_{m_2=0}^{\pm n}\Psi(x_L, x_U, y_L, y_U, L_1^\beta, L_2^\beta), \tag{5.25}$$

where

$$\begin{aligned}
\Phi(x_L, x_U, y_L, y_U, p, q) = {} & \\
& + \frac{(qx_L + 2x_Ly_L + py_L - qx_U)\,\mathrm{sgn}\,(q+2y_L)\mathrm{sgn}\,(p+2x_L)}{4L_1^\alpha L_2^\alpha} \\
& - \frac{(qx_L + 2x_Ly_L + py_U - qx_U)\,\mathrm{sgn}\,(q+2y_U)\mathrm{sgn}\,(p+2x_U)}{4L_1^\alpha L_2^\alpha} \\
& + \frac{(p+2x_L)\left\{y_U \arctan\left(\frac{p+2x_L}{q+2y_U}\right) - y_L \arctan\left(\frac{p+2x_L}{q+2y_L}\right)\right\}}{2\pi L_1^\alpha L_2^\alpha} \\
& + \frac{(p+2x_U)\left\{y_U \arctan\left(\frac{q+2y_U}{p+2x_U}\right) - y_L \arctan\left(\frac{q+2y_L}{p+2x_U}\right)\right\}}{2\pi L_1^\alpha L_2^\alpha} \\
& + \frac{q(p+2x_L)\left\{\arctan\left(\frac{q+2y_L}{p+2x_L}\right) - \arctan\left(\frac{q+2y_U}{p+2x_L}\right)\right\}}{4\pi L_1^\alpha L_2^\alpha} \\
& + \frac{q(p+2x_U)\left\{\arctan\left(\frac{q+2y_U}{p+2x_U}\right) - \arctan\left(\frac{q+2y_L}{p+2x_U}\right)\right\}}{4\pi L_1^\alpha L_2^\alpha} \\
& - \frac{\{(p+2x_L)^2 - (q+2y_L)^2\}\ln\{(p+2x_L)^2 + (q+2y_L)^2\}}{16\pi L_1^\alpha L_2^\alpha} \\
& + \frac{\{(p+2x_L)^2 - (q+2y_U)^2\}\ln\{(p+2x_L)^2 + (q+2y_U)^2\}}{16\pi L_1^\alpha L_2^\alpha} \\
& - \frac{\{(p+2x_U)^2 - (q+2y_U)^2\}\ln\{(p+2x_U)^2 + (q+2y_U)^2\}}{16\pi L_1^\alpha L_2^\alpha} \\
& + \frac{\{(p+2x_U)^2 - (q+2y_L)^2\}\ln\{(p+2x_U)^2 + (q+2y_L)^2\}}{16\pi L_1^\alpha L_2^\alpha}
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
\Psi(x_L, x_U, y_L, y_U, p, q) = {} & (32\pi L_1^\alpha L_2^\alpha)^{-1} \\
& \times [-\{(p+2x_L)^2 + (q+2y_L)^2\}\ln\{(p+2x_L)^2 + (q+2y_L)^2\} \\
& + \{(p+2x_L)^2 + (q-2y_L)^2\}\ln\{(p+2x_L)^2 + (q-2y_L)^2\} \\
& - \{(p+2x_L)^2 + (q-2y_U)^2\}\ln\{(p+2x_L)^2 + (q-2y_U)^2\}
\end{aligned}$$

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$$
\begin{aligned}
&+\{(p+2x_L)^2+(q+2y_U)^2\}\ln\{(p+2x_L)^2+(q+2y_U)^2\}\\
&-\{(p-2x_L)^2+(q-2y_L)^2\}\ln\{(p-2x_L)^2+(q-2y_L)^2\}\\
&+\{(p-2x_L)^2+(q-2y_U)^2\}\ln\{(p-2x_L)^2+(q-2y_U)^2\}\\
&-\{(p-2x_L)^2+(q+2y_U)^2\}\ln\{(p-2x_L)^2+(q+2y_U)^2\}\\
&+\{(p+2x_U)^2+(q+2y_L)^2\}\ln\{(p+2x_U)^2+(q+2y_L)^2\}\\
&-\{(p+2x_U)^2+(q-2y_L)^2\}\ln\{(p+2x_U)^2+(q-2y_L)^2\}\\
&+\{(p+2x_U)^2+(q-2y_U)^2\}\ln\{(p+2x_U)^2+(q-2y_U)^2\}\\
&-\{(p+2x_U)^2+(q+2y_U)^2\}\ln\{(p+2x_U)^2+(q+2y_U)^2\}\\
&+\{(p-2x_U)^2+(q-2y_L)^2\}\ln\{(p-2x_U)^2+(q-2y_L)^2\}\\
&-\{(p-2x_U)^2+(q+2y_L)^2\}\ln\{(p-2x_U)^2+(q+2y_L)^2\}\\
&+\{(p-2x_U)^2+(q+2y_U)^2\}\ln\{(p-2x_U)^2+(q+2y_U)^2\}\\
&-\{(p-2x_U)^2+(q-2y_U)^2\}\ln\{(p-2x_U)^2+(q-2y_U)^2\}].
\end{aligned}
\tag{5.27}
$$

6. Homogenized moduli

The homogenized elasticity tensor for the periodic composite material can be obtained by volume averaging equation (3.9) over the unit cell by means of equation (4.8). Noting that $\langle \Delta\sigma_{ij}(\boldsymbol{r})\rangle = \Delta\sigma^0_{ij}$ and $\langle \Delta\varepsilon^{\mathrm{T}}_{kl}(\boldsymbol{r})\rangle = \Delta\varepsilon^0_{kl}$, the homogenized constitutive relation is

$$
\Delta\sigma^0_{ij} = D^{\mathrm{m}}_{ijkl}\Delta\varepsilon^0_{kl} + \sum_{\beta=1}^{M} f^\beta \delta D^\beta_{ijkl}\Delta\varepsilon^{\mathrm{T}\beta}_{kl}. \tag{6.1}
$$

If the inverse matrix to $B^{\alpha\beta}_{klrs}$ in equations (4.14) and (4.15) is denoted by $A^{\alpha\beta}_{klrs}$, then we may write the solution to equation (4.14) in the form

$$
\Delta\varepsilon^{\mathrm{T}\beta}_{kl} = \sum_{\gamma=1}^{M} A^{\beta\gamma}_{klrs}\Delta\varepsilon^0_{rs} = M^\beta_{klrs}\Delta\varepsilon^0_{rs} \quad \text{for} \quad \beta = 1,2,\ldots,M, \tag{6.2}
$$

where

$$
M^\beta_{klrs} = \sum_{\gamma=1}^{M} A^{\beta\gamma}_{klrs} \quad \text{for} \quad 1 \le \beta \le M \tag{6.3}
$$

is the magnification or strain concentration tensor (Walker *et al.* 1991; Dvorak 1990, 1992; Dvorak & Benveniste 1992) which determines the local strain field at any subvolume in the fibre in terms of the uniform overall applied strain field. The strain magnification tensor for the matrix subvolumes,

$$
M^\alpha_{klrs} = I_{klrs} - \sum_{\beta=1}^{M}\sum_{\nu=1}^{M} f^\beta S^{\alpha\beta}_{klpq}A^{\beta\nu}_{pqrs} + \sum_{\gamma=1}^{N}\sum_{\beta=1}^{M}\sum_{\nu=1}^{M} f^\gamma f^\beta S^{\gamma\beta}_{klpq}A^{\beta\nu}_{pqrs}, \tag{6.4}
$$

for $M < \alpha \le N$, is obtained by substituting (6.2) into the integral equation (4.11). On substituting (6.2) into (6.1) we observe that the homogenized constitutive

relation can be written in the form

$$\Delta\sigma^0_{ij} = D^0_{ijkl}\Delta\varepsilon^0_{kl}, \tag{6.5}$$

where the homogenized or effective elasticity tensor is given by

$$D^0_{ijkl} = D^{\mathrm{m}}_{ijkl} + \sum_{\beta=1}^{M} f^\beta \delta D^\beta_{ijrs} M^\beta_{rskl}. \tag{6.6}$$

7. Numerical examples

The generalized self-consistent method, corresponding to equation (4.6) and figure 2*a*, is compared with higher order approximations involving $S_{kl}(\boldsymbol{r}, 1)$ and $S_{kl}(\boldsymbol{r}, 2)$ in figure 2*b* and *c*, for two separate composite materials. For three of the volume fractions, the unit cell is discretized into 49 square subvolumes and the local transverse stress field, assumed constant in each subvolume, is tabulated. At the largest volume fraction investigated, namely, 75.11%, the unit cell is discretized into 225 square subvolumes. The first composite material consists of a copper matrix containing periodically distributed cylindrical voids of square cross section, and computations are carried out for void volume fractions of 2.04%, 18.37%, 51.02% and 75.11%. In the second composite material the voids in the preceding computations are replaced by tungsten fibres of identical shape and volume fraction. This case determines the accuracy of the generalized self-consistent method when the fibres are stiffer than the matrix constituent phase.

The copper matrix with periodic voids is loaded in the transverse direction under a uniform overall applied stress of $\sigma^0_{11} = 1000$ kPa. All other components of σ^0_{ij} other than σ^0_{11} are assumed to be zero. The elastic moduli are assumed to be given by $E_{\mathrm{Cu}} = 127$ GPa and $\nu_{\mathrm{Cu}} = 0.34$. Figure 3*a*–*c* presents a numerical tabulation of the constant valued stresses, σ_{11}, for each of the 49 square subvolumes in the unit cell for void volume fractions of 2.04%, 18.37%, 51.02% and for each of the 225 subvolumes for the void volume fraction of 75.11% in figure 3*d*. In each subvolume the upper number represents the transverse stress according to the generalized self-consistent method ($n = 0$, figure 2*a*), while the numbers below represent the transverse stress when the unit cell is surrounded by its eight nearest neighbours ($n = 1$, figure 2*b*) and an additional 16 surrounding neighbours ($n = 2$, figure 2*c*), respectively. As may be observed from these tabulations, the case where $n = 1$ has converged to within an accuracy of four significant figures, and as such, can be used to construct error estimates of the generalized self-consistent method, where $n = 0$.

When the void occupies only a 2.04% volume fraction of the unit cell, the results in figure 3*a* show that the local transverse stress field of the generalized self-consistent method is accurate to within 3% throughout the copper matrix phase. Table 1 shows that the generalized self-consistent transverse Young's modulus is accurate to within 0.1% at this low volume fraction.

At a volume fraction of 18.37% the local transverse stress field is in error by up to 35% in the matrix ligaments perpendicular to the applied stress. However, the transverse stresses in the ligaments are small compared with those in the ligaments parallel to the applied stress, and contribute but a small error to the homogenized transverse modulus. The error in the homogenized modulus is

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(*a*)

1032 1058 1058	1045 1063 1063	1045 1055 1055	1038 1044 1044	1045 1055 1055	1045 1063 1063	1032 1058 1058
1018 1027 1027	1049 1052 1052	1084 1082 1082	1075 1073 1073	1084 1082 1082	1049 1052 1052	1018 1027 1027
973 958 958	974 965 965	1095 1084 1084	1397 1382 1382	1095 1084 1084	974 965 965	973 958 958
938 910 910	852 838 838	559 553 553	0 0 0	559 553 553	852 838 838	938 910 910
973 958 958	974 965 965	1095 1084 1084	1397 1382 1382	1095 1084 1084	974 965 965	973 958 958
1018 1027 1027	1049 1052 1052	1084 1082 1082	1075 1073 1073	1084 1082 1082	1049 1052 1052	1018 1027 1027
1032 1058 1058	1045 1063 1063	1045 1055 1055	1038 1044 1044	1045 1055 1055	1045 1063 1063	1032 1058 1058

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

(*b*)

1331 1503 1504	1450 1554 1555	1516 1585 1585	1566 1627 1627	1516 1585 1585	1450 1554 1555	1331 1503 1504
1156 1201 1201	1425 1407 1407	2001 1914 1914	1968 1872 1872	2001 1914 1914	1425 1407 1407	1156 1201 1201
737 626 626	504 463 463	0 0 0	0 0 0	0 0 0	504 463 463	737 626 626
520 337 336	196 148 148	0 0 0	0 0 0	0 0 0	196 148 148	520 337 336
737 626 626	504 463 463	0 0 0	0 0 0	0 0 0	504 463 463	737 626 626
1156 1201 1201	1425 1407 1407	2001 1914 1914	1968 1872 1872	2001 1914 1914	1425 1407 1407	1156 1201 1201
1331 1503 1504	1450 1554 1555	1516 1585 1585	1566 1627 1627	1516 1585 1585	1450 1554 1555	1331 1503 1504

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

(*c*)

2509 2730 2732	3571 3499 3499	3315 3500 3500	3207 3500 3500	3315 3500 3500	3571 3499 3499	2509 2730 2732
964 745 745	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	964 745 745
227 39 39	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	227 39 39
115 −32 −32	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	115 −32 −32
227 39 39	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	227 39 39
964 745 745	0 0 0	0 0 0	0 0 0	0 0 0	0 0 0	964 745 745
2509 2730 2732	3571 3499 3499	3315 3500 3500	3207 3500 3500	3315 3500 3500	3571 3499 3499	2509 2730 2732

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

Figure 3. Local elastic transverse stress field, σ_{11}, in a voided periodic composite subjected to a uniform overall applied stress of $\sigma_{11}^0 = 1000$ kPa: (*a*) void volume fraction = 2.04%; (*b*) void volume fraction = 18.37%; (*c*) void volume fraction = 51.02%.

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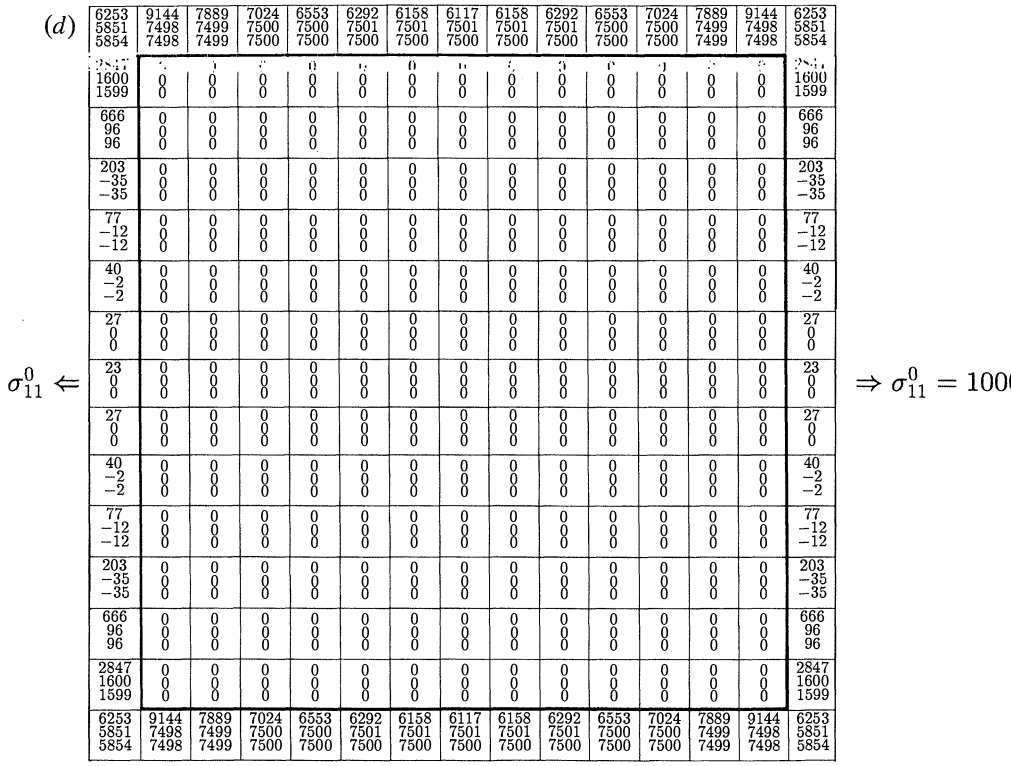

Figure 3. Local elastic transverse stress field, σ_{11}, in a voided periodic composite subjected to a uniform overall applied stress of $\sigma_{11}^0 = 1000$ kPa: (*d*) void volume fraction = 75.11%.

Table 1. *Homogenized transverse Young's modulus for copper with periodically distributed voids*

void volume fraction		2.04%	18.37%	51.02%	75.11%
approximation		transverse Young's modulus/GPa			
generalized self-consistent method	$(n = 0)$	120.54	80.48	34.84	14.21
with 8 nearest neighbours included	$(n = 1)$	120.63	83.49	40.47	18.39
with 24 nearest neighbours included	$(n = 2)$	120.63	83.50	40.48	18.40

further reduced because of the volume averaging. For example, the stresses in the ligaments parallel to the applied stress are underpredicted by the generalized self-consistent method, while those in the ligaments perpendicular to the applied stress are overpredicted. This cancelling of errors is due to equilibrium considerations, which dictate that the average of the local stress field must equal the applied stress field. Table 1 shows that the overall transverse modulus is accurate to within 4% at 18.37% void volume fraction.

When the void volume fraction increases to 51.02% the transverse modulus in table 1 is in error by 14%. The middle of the perpendicular ligament is now in compression at a transverse stress of −32 kPa, while the generalized self-consistent method predicts a tensile stress of 115 kPa. At the largest void volume fraction of 75.11% the error in the transverse modulus rises to 23%. Large errors

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Table 2. *Homogenized transverse Young's modulus for tungsten fibre–copper matrix periodic composite*

fibre volume fraction		2.04%	18.37%	51.02%	75.11%
approximation		transverse Young's modulus/GPa			
generalized self-consistent method	$(n = 0)$	129.85	154.69	220.89	291.92
with 8 nearest neighbours included	$(n = 1)$	129.87	156.18	229.07	300.68
with 24 nearest neighbours included	$(n = 2)$	129.87	156.18	229.09	300.70

are again observed in the perpendicular ligaments. However, the transverse stress is zero in the central portions of the perpendicular ligaments and the subvolumes in this region, which possess the largest error in the transverse stress, contribute but little to the homogenized modulus compared with the subvolumes with large transverse stresses in the ligaments parallel to the applied stress.

An examination of figure 3 *a–d* shows that the effect of including the eight nearest neighbours gives a very accurate representation of the local transverse stress field within the unit cell. The effect of adding an additional layer of unit cells comprised of the next 16 nearest neighbours results in changes of less than 0.1% to the local stress field.

In the second example we use the same loading condition but now look at the case where the fibre is stiffer than the copper matrix phase of the unit cell. In particular, we examine the local transverse stress field and homogenized transverse Young's modulus for a tungsten–copper fibrous composite. The tungsten fibre is of square planform and occupies the same volume fractions as the preceding voids. The elastic moduli of the isotropic tungsten phase are assumed to be given by $E_{\mathrm{W}} = 395$ GPa and $\nu_{\mathrm{W}} = 0.28$. Transverse stresses are depicted in figure 4 *a–d*, and the homogenized transverse Young's moduli are listed in table 2.

A general conclusion is that when the fibre phase is stiffer than the matrix phase by about a factor of three, the generalized self-consistent method gives results which are more accurate than the extreme case of voids. Errors in the local stress field are now largest in the matrix ligaments parallel to the applied stress field, and are smallest in the perpendicular ligaments, which contrasts with the behaviour in the voided composite. In both examples the generalized self-consistent method underestimates the effective transverse Young's modulus of the composite.

In figure 4 *e* the results from the Fourier series approximation (cf. (2.5)) given in Walker *et al.* (1991, 1992) can be compared with the Green's function results in figure 4 *b*. The Fourier series computations in figure 4 *e* show the results of summing the Fourier series to $n_p = \pm 50$, ± 70 and ± 200 in equation (2.5). Agreement between the results in figure 4 *b* and *e* is very good and demonstrates the numerical equivalency of the Poisson sum transformation of the Fourier series into the equivalent Green's function method. The rapid convergence of the Green's function Poisson sum in comparison with the Fourier series representation is especially noteworthy.

The two preceding examples compare the accuracy of the generalized self-

(a)

989 977 977	985 977 977	986 983 983	990 989 989	986 983 983	985 977 977	989 977 977
993 989 989	982 980 980	972 972 972	980 981 981	972 972 972	982 980 980	993 989 989
1010 1015 1015	1008 1012 1012	963 966 966	874 877 877	963 966 966	1008 1012 1012	1010 1015 1015
1023 1024 1024	1053 1060 1060	1148 1154 1154	1295 1302 1302	1148 1154 1154	1053 1060 1060	1023 1024 1024
1010 1015 1015	1008 1012 1012	963 966 966	874 877 877	963 966 966	1008 1012 1012	1010 1015 1015
993 989 989	982 980 980	972 972 972	980 981 981	972 972 972	982 980 980	993 989 989
989 977 977	985 977 977	986 983 983	990 989 989	986 983 983	985 977 977	989 977 977

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

(b)

914 825 825	883 820 819	874 836 836	866 838 838	874 836 836	883 820 819	914 825 825
967 937 937	895 877 877	751 741 741	696 689 689	751 741 741	895 877 877	967 937 937
1081 1131 1131	1154 1193 1193	1296 1337 1337	1350 1393 1393	1296 1337 1337	1154 1193 1193	1081 1131 1131
1124 1209 1210	1155 1217 1217	1115 1169 1169	1104 1156 1157	1115 1169 1169	1155 1217 1217	1124 1209 1210
1081 1131 1131	1154 1193 1193	1296 1337 1337	1350 1393 1393	1296 1337 1337	1154 1193 1193	1081 1131 1131
967 937 937	895 877 877	751 741 741	696 689 689	751 741 741	895 877 877	967 937 937
914 825 825	883 820 819	874 836 836	866 838 838	874 836 836	883 820 819	914 825 825

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

(c)

820 677 676	682 577 577	609 502 502	592 477 476	609 502 502	682 577 577	820 677 676
1079 1067 1067	1232 1200 1199	1314 1278 1278	1335 1298 1298	1314 1278 1278	1232 1200 1199	1079 1067 1067
1071 1166 1167	1059 1146 1146	1067 1148 1149	1074 1154 1154	1067 1148 1149	1059 1146 1146	1071 1166 1167
1054 1177 1178	1033 1151 1151	1023 1139 1139	1023 1138 1138	1023 1139 1139	1033 1151 1151	1054 1177 1178
1071 1166 1167	1059 1146 1146	1067 1148 1149	1074 1154 1154	1067 1148 1149	1059 1146 1146	1071 1166 1167
1079 1067 1067	1232 1200 1199	1314 1278 1278	1335 1298 1298	1314 1278 1278	1232 1200 1199	1079 1067 1067
820 677 676	682 577 577	609 502 502	592 477 476	609 502 502	682 577 577	820 677 676

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

Figure 4. Local elastic transverse stress field, σ_{11}, in a tungsten–copper periodic composite subjected to a uniform overall applied stress of $\sigma_{11}^0 = 1000$ kPa: (*a*) fibre volume fraction = 2.04%; (*b*) fibre volume fraction = 18.37%; (*c*) fibre volume fraction = 51.02%.

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(*d*)

769 606 605	636 517 517	562 445 444	533 401 401	521 378 378	516 366 366	513 361 360	512 359 359	513 361 360	516 366 366	521 378 378	533 401 401	562 445 444	636 517 517	769 606 605
1042 946 945	1200 1061 1060	1295 1129 1128	1335 1150 1150	1354 1153 1152	1363 1150 1150	1367 1148 1147	1368 1146 1146	1367 1148 1147	1363 1150 1150	1354 1153 1152	1335 1150 1150	1295 1129 1128	1200 1061 1060	1042 946 945
1052 1055 1054	1066 1050 1050	1101 1066 1066	1140 1086 1085	1169 1099 1099	1188 1107 1107	1199 1111 1111	1202 1112 1112	1199 1111 1111	1188 1107 1107	1169 1099 1099	1140 1086 1085	1101 1066 1066	1066 1050 1050	1052 1055 1054
1026 1081 1080	1027 1073 1072	1037 1072 1072	1052 1078 1078	1068 1083 1083	1080 1088 1088	1087 1090 1090	1089 1091 1091	1087 1090 1090	1080 1088 1088	1068 1083 1083	1052 1078 1078	1037 1072 1072	1027 1073 1072	1026 1081 1080
1005 1088 1088	1001 1082 1082	1001 1078 1079	1006 1079 1079	1012 1081 1081	1018 1082 1082	1022 1083 1083	1023 1083 1084	1022 1083 1083	1018 1082 1082	1012 1081 1081	1006 1079 1079	1001 1078 1079	1001 1082 1082	1005 1088 1088
990 1089 1089	984 1085 1085	979 1081 1082	979 1080 1081	981 1081 1081	983 1081 1081	985 1081 1082	986 1081 1082	985 1081 1082	983 1081 1081	981 1081 1081	979 1080 1081	979 1082 1082	984 1085 1085	990 1089 1089
982 1089 1090	974 1086 1087	968 1083 1084	965 1081 1082	965 1081 1081	966 1081 1082	967 1081 1082	967 1081 1082	967 1081 1082	966 1081 1082	965 1081 1081	965 1081 1082	968 1083 1084	974 1086 1087	982 1089 1090
979 1089 1089	971 1086 1086	965 1083 1083	961 1082 1081	960 1081 1081	960 1081 1081	961 1081 1081	961 1081 1081	961 1081 1081	960 1081 1081	960 1081 1081	961 1082 1081	965 1083 1083	971 1086 1086	979 1089 1089
982 1089 1090	974 1086 1087	968 1083 1084	965 1081 1082	965 1081 1081	966 1081 1082	967 1081 1082	967 1081 1082	967 1081 1082	966 1081 1082	965 1081 1081	965 1081 1082	968 1083 1084	974 1086 1087	982 1089 1090
990 1089 1089	984 1085 1085	979 1081 1082	979 1080 1081	981 1081 1081	983 1081 1081	985 1081 1082	986 1081 1082	985 1081 1082	983 1081 1081	981 1081 1081	979 1080 1081	979 1082 1082	984 1085 1085	990 1089 1089
1005 1088 1088	1001 1082 1082	1001 1078 1079	1006 1079 1079	1012 1081 1081	1018 1082 1082	1022 1083 1083	1023 1083 1084	1022 1083 1083	1018 1082 1082	1012 1081 1081	1006 1079 1079	1001 1078 1079	1001 1082 1082	1005 1088 1088
1026 1081 1080	1027 1073 1072	1037 1072 1072	1052 1078 1078	1068 1083 1083	1080 1088 1088	1087 1090 1090	1089 1091 1091	1087 1090 1090	1080 1088 1088	1068 1083 1083	1052 1078 1078	1037 1072 1072	1027 1073 1072	1026 1081 1080
1052 1055 1054	1066 1050 1050	1101 1066 1066	1140 1086 1085	1169 1099 1099	1188 1107 1107	1199 1111 1111	1202 1112 1112	1199 1111 1111	1188 1107 1107	1169 1099 1099	1140 1086 1085	1101 1066 1066	1066 1050 1050	1052 1055 1054
1042 946 945	1200 1061 1060	1295 1129 1128	1335 1150 1150	1354 1153 1152	1363 1150 1150	1367 1148 1147	1368 1146 1146	1367 1148 1147	1363 1150 1150	1354 1153 1152	1335 1150 1150	1295 1129 1128	1200 1061 1060	1042 946 945
769 606 605	636 517 517	562 445 444	533 401 401	521 378 378	516 366 366	513 361 360	512 359 359	513 361 360	516 366 366	521 378 378	533 401 401	562 445 444	636 517 517	769 606 605

$\sigma^0_{11} \Leftarrow$ $\Rightarrow \sigma^0_{11} = 1000$

(*e*)

825 825 825	819 819 819	835 835 835	837 837 838	835 835 835	819 819 819	825 825 825
937 937 937	877 877 877	740 740 741	688 689 689	740 740 741	877 877 877	937 937 937
1132 1132 1132	1188 1189 1192	1344 1342 1339	1394 1394 1394	1344 1342 1339	1188 1189 1192	1132 1132 1132
1211 1210 1210	1212 1213 1216	1175 1173 1171	1156 1157 1157	1175 1173 1171	1212 1213 1216	1211 1210 1210
1132 1132 1132	1188 1189 1192	1344 1342 1339	1394 1394 1394	1344 1342 1339	1188 1189 1192	1132 1132 1132
937 937 937	877 877 877	740 740 741	688 689 689	740 740 741	877 877 877	937 937 937
825 825 825	819 819 819	835 835 835	837 837 838	835 835 835	819 819 819	825 825 825

$\sigma^0_{11} \Leftarrow$ $\Rightarrow \sigma^0_{11} = 1000$

Figure 4. Local elastic transverse stress field, σ_{11}, in a tungsten–copper periodic composite subjected to a uniform overall applied stress of $\sigma^0_{11} = 1000$ kPa: (*d*) fibre volume fraction = 75.11%; (*e*) fibre volume fraction = 18.37%, Fourier series solution.

consistent method for a two-dimensional fibrous medium, when the integral equation governing the distribution of the local strain field in the unit cell is solved numerically. We now examine the case of a one-dimensional laminated medium in which slabs of tungsten are sandwiched between slabs of copper. This allows the numerical results to be compared with an exact solution. In figure 5 the unit cell is discretized into 49 subvolumes and is embedded in the homogenized medium

A

980	993	1039	1110	1039	993	980
997	1005	999	973	999	1005	997
1000	996	982	969	982	996	1000
999	993	978	967	978	993	999
1000	996	982	969	982	996	1000
997	1005	999	973	999	1005	997
980	993	1039	1110	1039	993	980

$\sigma_{11}^0 \Leftarrow$ $\Rightarrow \sigma_{11}^0 = 1000$

B

Figure 5. Generalized self-consistent approximation ($n = 0$) to the local elastic transverse stress field, σ_{11}, in a one-dimensional laminated medium.

corresponding to the generalized self-consistent approximation, $S_{kl}(\boldsymbol{r}, 0)$. The tungsten occupies the seven subvolumes between the points A and B in the unit cell of figure 5, and when this cell is repeated in both directions the overall material corresponds to a one-dimensional laminated structure. When this medium is loaded with an overall transverse stress of $\sigma_{11}^0 = 1000$ kPa, with all the other components being zero, the transverse stress in the medium must be uniform and equal to the overall applied stress.

The transverse stress distribution in the unit cell according to the generalized self-consistent method ($n = 0$) is shown in figure 5. In the actual laminated structure, and in the higher order approximations $S_{kl}(\boldsymbol{r}, 1)$ and $S_{kl}(\boldsymbol{r}, 2)$, the subvolumes at points A and B interact with adjacent tungsten subvolumes in the surrounding neighbours, and the transverse stress distribution in the nearest neighbour approximation ($n = 1$) is within 0.1% of 1000 kPa for all the subvolumes in the unit cell. However, in the generalized self-consistent approximation, the tungsten subvolumes at points A and B interact with the surrounding homogenized medium. Consequently, the stress is higher in these subvolumes and lower in the remaining tungsten subvolumes. The formation of a ridged valley (Walker *et al.* 1991) in the tungsten in figure 5 is apparent, where the transverse stress forms a central valley parallel to the loading direction in which the stress is approximately 967 kPa, with values of 1110 kPa in the north and south ridges at A and B. In this discretization the generalized self-consistent method gives a maximum subvolume stress error of 11% within the unit cell.

An exact representation of the transverse Young's modulus for a laminated medium can be obtained if we assume that the strains in the directions x_2, x_3 transverse to the loading direction are equal in the two phases, and that the only overall stress component is σ_{11}^0. If the volume fractions of the fibre and matrix phases are f and $1 - f$, respectively, then the overall transverse Young's modulus

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Table 3. *Comparison of transverse Young's modulus for tungsten fibre–copper matrix one-dimensional laminated and two-dimensional fibrous composite*

transverse Young's modulus/GPa		
one-dimensional laminated lattice	two-dimensional fibrous lattice	tungsten volume fraction
130.15	129.87	2.04%
155.84	156.18	18.37%
217.30	229.09	51.02%
283.38	300.70	75.11%

can be written in the form,

$$E_{\mathrm{T}} = \left[\frac{f}{E_{\mathrm{f}}} + \frac{1-f}{E_{\mathrm{m}}} - \frac{2f\left(\dfrac{\nu_{\mathrm{f}}}{E_{\mathrm{f}}} - \dfrac{\nu_{\mathrm{m}}}{E_{\mathrm{m}}}\right)^2}{\left\{\dfrac{1-\nu_{\mathrm{f}}}{E_{\mathrm{f}}} + \left(\dfrac{f}{1-f}\right)\left(\dfrac{1-\nu_{\mathrm{m}}}{E_{\mathrm{m}}}\right)\right\}} \right]^{-1}. \tag{7.1}$$

This yields a value of $E_{\mathrm{T}} = 149.27$ GPa for the tungsten/copper laminated material, which may be compared with the generalized self-consistent value ($n = 0$) in figure 5 of $E_{\mathrm{T}}^{n=0} = 147.43$ GPa. The next order approximation ($n = 1$) gives $E_{\mathrm{T}}^{n=1} = 149.27$ GPa, which is correct to two decimal places (five significant figures).

The approximation $S_{kl}(\boldsymbol{r}, 1)$, which includes the interaction of the unit cell with its eight nearest neighbours, gives results for a laminated medium which are accurate to within 0.1% for both the moduli and the local elastic stress field.

A final observation which may be adduced from the computations is the relative insensitivity of the homogenized transverse modulus to the shape of the fibrous inclusions. This insensitivity has also been noted by Zhao & Weng (1990). In table 3 we exhibit the transverse modulus for the two-dimensional fibrous medium compared with the one-dimensional laminated medium for the same tungsten fibre volume fractions. Even when the volume fraction increases to 75% the difference in the transverse Young's modulus in the two materials amounts to only 6%.

8. Conclusions

The effects of fibres distributed in periodic fashion in an infinite uniform matrix phase can be assimilated to the summation of fictitious periodic body forces in the uniform matrix. Local elastic fields within the unit cell can be obtained by summing the effects of these body forces from the infinite lattice of fibres with a Green's function approach, and by ensuring that the volume averages of the local elastic fields over a unit cell in the periodic lattice are equal to the uniform overall applied fields. This paper demonstrates that the generalized self-consistent method is equivalent to the zeroth term in the infinite body force summation, and by comparing the zeroth term with higher order sums representing the contribution from layers of neighbouring unit cells in the periodic lattice, the accuracy of

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the generalized self-consistent method has been assessed. The accuracy is found to be very good, even in the extreme case of a composite containing a high volume fraction of voids. When the interaction between the unit cell and its eight nearest neighbours is included, the integral equation gives results which are very near the exact results, and even a laminated medium can be accurately modelled. The effect of including the interaction between the unit cell and the sixteen additional neighbours surrounding the nearest eight is found to be very small, showing that the Poisson sum provides a rapidly convergent solution for the periodic composite material. A history of the need to subtract the volume averaged polarization term from the volume integral of a periodic boundary value problem to obtain absolutely convergent solutions is discussed.

The work of two of the authors (K.P.W. and E.H.J.) was supported by the United States Department of Energy under Grant Number DE–FG02–92ER14247. Dr Oscar P. Manley served as contract monitor.

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Received 11 August 1992; accepted 5 February 1993

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